# Chapter 1

## Trading Strategies

Suppose that some stock $S$ can be traded on $N$ trading days

$$t_1, t_2, \cdots, t_k, \cdots, t_{N-1}, t_N \tag{1.1}$$

and suppose that

$$S_k \;\; := \;\; S(t_k) \tag{1.2}$$

denotes the price of the stock on day $t_k$. Suppose start time, say, today, is $t_0$ and $S_0 := S(t_0)$ denotes the price of the stock at start time.

Suppose that at start time we have an initial cash amount of $V_0$. This amount lies on some bank account and interest rates will be paid on that amount. To fix notation, let us assume that we have a constant yearly interest rate of $r$ (so, for example, $r = 3\%$) and that the $t_k$ are expressed in year fraction (that is, if, say, $t_0 = 0$ is Jan 1st 2014 and $t_k$ is supposed to be Oct 1st, 2014, then approximately $t_k \approx 0.75$). Thus, if we simply put $V_0$ on a bank account and do nothing (that is, no buying or selling of stocks) then at time $t_k$ we have a cash amount $V_k$ given by

$$V_k \;\; = \;\; e^{r(t_k - t_0)} V_0 \tag{1.3}$$

Now, consider the following trading strategy:

- At start time $t_0$ buy a number of $\delta_0$ stocks.
- At time $t_1$, sell these $\delta_0$ stocks and buy a number of $\delta_1$ stocks. Or, more precisely, if $\delta_1 - \delta_0 \geq 0$, at time $t_1$ buy a number of $\delta_1 - \delta_0$ stocks, or, if $\delta_1 - \delta_0 < 0$, at time $t_1$ sell a number of $\delta_1 - \delta_0$ stocks. To put it in another way: At time $t_1$ readjust your stock position such that you hold a number of $\delta_1$ stocks at the end of day $t_1$.
- At time $t_2$, sell these $\delta_1$ stocks and buy a number of $\delta_2$ stocks such that your stock position is $\delta_2$ stocks at the end of day $t_2$.

$$\vdots$$

- At time $t_k$, sell $\delta_{k-1}$ stocks and buy a number of $\delta_k$ stocks such that your stock position is $\delta_k$ stocks at the end of day $t_k$.

$$\vdots \tag{1.4}$$

- At time $t_N$, sell $\delta_{N-1}$ stocks and buy no new stocks such that your stock position is closed at the end of day $t_N$.

Then this trading strategy has generated the following amount of money:

**Theorem 1.1:** Consider the trading strategy (1.4). Then at time $t_k$ this strategy has generated an amount $V_k$ given by the following expressions, $k = 1, 2, ..., N-1, N$:

**a)** If the interest rates are zero, $r = 0$, then

$$V_k = V_0 + \sum_{j=1}^{k} \delta_{j-1}(S_j - S_{j-1}) \tag{1.5}$$

**b)** If the interest rates are non zero, $r \neq 0$, then

$$\begin{aligned} V_k &= e^{r(t_k - t_0)} V_0 + \sum_{j=1}^{k} \delta_{j-1} \left( e^{r(t_k - t_j)} S_j - e^{r(t_k - t_{j-1})} S_{j-1} \right) \\ &= e^{r(t_k - t_0)} \Big\{ V_0 + \sum_{j=1}^{k} \delta_{j-1} \left( e^{-r(t_j - t_0)} S_j - e^{-r(t_{j-1} - t_0)} S_{j-1} \right) \Big\} \end{aligned} \tag{1.6}$$

This can be written more compactly if we define the discounted quantities

$$v_j := e^{-r(t_j - t_0)} V_j\,, \quad j \in \{0, 1, ..., N\} \tag{1.7}$$

$$s_j := e^{-r(t_j - t_0)} S_j\,, \tag{1.8}$$

then (1.6) becomes

$$v_k = v_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1}) \tag{1.9}$$

**Proof:** At start time $t_0$ an amount of $\delta_0$ stocks shall be bought. Each stock costs $S_0$, thus we need a cash amount of $\delta_0 S_0$ of $V_0$ to buy $\delta_0$ stocks for the price $S_0$ each,

$$V_0 = \underbrace{(V_0 - \delta_0 S_0)}_{\text{money part}} + \underbrace{\delta_0 S_0}_{\text{stock part}} \tag{1.10}$$

Now time elapses from $t_0$ to $t_1$. The portfolio (1.10) has changed in value, since the stock price is now $S_1$ and on the cash part interest rate has been paid. The money part becomes

$$V_0 - \delta_0 S_0 \overset{t_0 \to t_1}{\longrightarrow} (V_0 - \delta_0 S_0)e^{r(t_1-t_0)} \tag{1.11}$$

and the stock part has value

$$\delta_0 S_0 \overset{t_0 \to t_1}{\longrightarrow} \delta_0 S_1 \tag{1.12}$$

such that the value of the whole portfolio at time $t_1$ is

$$V_1 = \underbrace{(V_0 - \delta_0 S_0)e^{r(t_1-t_0)}}_{\text{money part}} + \underbrace{\delta_0 S_1}_{\text{stock part}} = V_0\, e^{r(t_1-t_0)} + \delta_0(S_1 - S_0\, e^{r(t_1-t_0)}) \tag{1.13}$$

'At the end of time $t_1$', that is, right after the change in value from $S_0$ to $S_1$ but before the next change to $S_2$, the bank readjusts the portfolio. The number of stocks is changed from $\delta_0$ to $\delta_1$. Thus the money part increases or decreases by the amount $\delta_0 S_1 - \delta_1 S_1$ such that

$$\begin{aligned} V_1 &= (V_0 - \delta_0 S_0)e^{r(t_1-t_0)} + \delta_0 S_1 - \delta_1 S_1 \; + \; \delta_1 S_1 \\ &= \underbrace{V_0\, e^{r(t_1-t_0)} + \delta_0(S_1 - S_0\, e^{r(t_1-t_0)}) - \delta_1 S_1}_{\text{money part}} + \underbrace{\delta_1 S_1}_{\text{stock part}} \end{aligned} \tag{1.14}$$

Now time elapses from $t_1$ to $t_2$ and the stock has changed in value, thus also the portfolio has changed value. It becomes

$$\begin{aligned} V_2 &= \underbrace{\big(V_0\, e^{r(t_1-t_0)} + \delta_0(S_1 - S_0\, e^{r(t_1-t_0)}) - \delta_1 S_1\big)e^{r(t_2-t_1)}}_{\text{money part}} + \underbrace{\delta_1 S_2}_{\text{stock part}} \\ &= \underbrace{V_0\, e^{r(t_2-t_0)} + \delta_0(S_1\, e^{r(t_2-t_1)} - S_0\, e^{r(t_2-t_0)}) - \delta_1 S_1\, e^{r(t_2-t_1)}}_{\text{money part}} + \underbrace{\delta_1 S_2}_{\text{stock part}} \qquad (1.15) \\ &= V_0\, e^{r(t_2-t_0)} + \delta_0(S_1\, e^{r(t_2-t_1)} - S_0\, e^{r(t_2-t_0)}) + \delta_1(S_2 - S_1\, e^{r(t_2-t_1)}) \qquad (1.16) \end{aligned}$$

'At the end of time $t_2$' the bank changes the number of stocks in the portfolio from $\delta_1$ to $\delta_2$. Thus from (1.15) we get

$$\begin{aligned} V_2 &= V_0\, e^{r(t_2-t_0)} + \delta_0(S_1\, e^{r(t_2-t_1)} - S_0\, e^{r(t_2-t_0)}) - \delta_1 S_1\, e^{r(t_2-t_1)} + \delta_1 S_2 - \delta_2 S_2 \; + \; \delta_2 S_2 \\ &= \underbrace{V_0\, e^{r(t_2-t_0)} + \delta_0(S_1\, e^{r(t_2-t_1)} - S_0\, e^{r(t_2-t_0)}) + \delta_1(S_2 - S_1\, e^{r(t_2-t_1)}) - \delta_2 S_2}_{\text{money part}} + \underbrace{\delta_2 S_2}_{\text{stock part}} \end{aligned}$$

Continuing in this way up to time $t_k$, the value of the portfolio seems to be

$$\begin{aligned} V_k &= V_0\, e^{r(t_k-t_0)} + \delta_0(S_1\, e^{r(t_k-t_1)} - S_0\, e^{r(t_k-t_0)}) \\ &\quad + \delta_1(S_2\, e^{r(t_k-t_2)} - S_1\, e^{r(t_k-t_1)}) \\ &\quad + \cdots \\ &\quad + \delta_{k-1}(S_k - S_{k-1}\, e^{r(t_k-t_{k-1})}) \\ &= e^{r(t_k-t_0)}\Big\{V_0 + \sum_{j=1}^{k} \delta_{j-1}(S_j\, e^{-r(t_j-t_0)} - S_{j-1}\, e^{-r(t_{j-1}-t_0)})\Big\} \end{aligned} \tag{1.17}$$

and let us proof this by induction on $k$: For $k = 1$, we verified this already. Suppose (1.17) holds for $k$. At $t_k$, we are holding $\delta_{k-1}$ stocks. To prepare for day $t_{k+1}$, we readjust and sell $\delta_{k-1}$ stocks and buy $\delta_k$ stocks to hold $\delta_k$ stocks at the end of day $t_k$. Before doing this, the stock part of $V_k$ is $\delta_{k-1}S_k$ and the money part is

$$\begin{aligned} V_0\,e^{r(t_k-t_0)} &+ \delta_0(S_1\,e^{r(t_k-t_1)} - S_0\,e^{r(t_k-t_0)}) \\ &+ \delta_1(S_2\,e^{r(t_k-t_2)} - S_1\,e^{r(t_k-t_1)}) \\ &+ +\cdots \\ &+ \delta_{k-2}(S_{k-1}\,e^{r(t_k-t_{k-1})} - S_{k-2}\,e^{r(t_k-t_{k-2})}) \\ &- \delta_{k-1}S_{k-1}\,e^{r(t_k-t_{k-1})} \end{aligned} \tag{1.18}$$

and after the readjustment, still at time $t_k$, the stock part is $\delta_k S_k$ and the money part is

$$\begin{aligned} V_0\,e^{r(t_k-t_0)} &+ \delta_0(S_1\,e^{r(t_k-t_1)} - S_0\,e^{r(t_k-t_0)}) \\ &+ \delta_1(S_2\,e^{r(t_k-t_2)} - S_1\,e^{r(t_k-t_1)}) \\ &+ \cdots \\ &+ \delta_{k-2}(S_{k-1}\,e^{r(t_k-t_{k-1})} - S_{k-2}\,e^{r(t_k-t_{k-2})}) \\ &- \delta_{k-1}S_{k-1}\,e^{r(t_k-t_{k-1})} + \delta_{k-1}S_k - \delta_k S_k \end{aligned} \tag{1.19}$$

Now time elapses from $t_k$ to $t_{k+1}$. Interest rates have been paid on the cash part and the stock has changed in value. The cash part has become

$$\begin{aligned} e^{r(t_{k+1}-t_k)}\Big\{V_0\,e^{r(t_k-t_0)} &+ \delta_0(S_1\,e^{r(t_k-t_1)} - S_0\,e^{r(t_k-t_0)}) \\ &+ \delta_1(S_2\,e^{r(t_k-t_2)} - S_1\,e^{r(t_k-t_1)}) \\ &+ \cdots \\ &+ \delta_{k-2}(S_{k-1}\,e^{r(t_k-t_{k-1})} - S_{k-2}\,e^{r(t_k-t_{k-2})}) \\ &+ \delta_{k-1}(S_k - S_{k-1}\,e^{r(t_k-t_{k-1})} - \delta_k S_k\Big\} \\ = V_0\,e^{r(t_{k+1}-t_0)} &+ \delta_0(S_1\,e^{r(t_{k+1}-t_1)} - S_0\,e^{r(t_{k+1}-t_0)}) \\ &+ \delta_1(S_2\,e^{r(t_{k+1}-t_2)} - S_1\,e^{r(t_{k+1}-t_1)}) \\ &+ \cdots \\ &+ \delta_{k-2}(S_{k-1}\,e^{r(t_{k+1}-t_{k-1})} - S_{k-2}\,e^{r(t_{k+1}-t_{k-2})}) \\ &+ \delta_{k-1}(S_k\,e^{r(t_{k+1}-t_k)} - S_{k-1}\,e^{r(t_{k+1}-t_{k-1})}) - \delta_k S_k\,e^{r(t_{k+1}-t_k)} \end{aligned} \tag{1.20}$$

and the stock part has become

$$\delta_k\,S_{k+1} \tag{1.21}$$

Thus, the total portfolio value at time $t_{k+1}$ is

$$\begin{aligned} V_{k+1} &= V_0\, e^{r(t_{k+1}-t_0)} \;+\; \delta_0\big(S_1\, e^{r(t_{k+1}-t_1)} - S_0\, e^{r(t_{k+1}-t_0)}\big) \\ &\quad + \delta_1\big(S_2\, e^{r(t_{k+1}-t_2)} - S_1\, e^{r(t_{k+1}-t_1)}\big) \\ &\quad + \cdots \\ &\quad + \delta_{k-2}\big(S_{k-1}\, e^{r(t_{k+1}-t_{k-1})} - S_{k-2}\, e^{r(t_{k+1}-t_{k-2})}\big) \\ &\quad + \delta_{k-1}\big(S_k\, e^{r(t_{k+1}-t_k)} - S_{k-1}\, e^{r(t_{k+1}-t_{k-1})} \;-\; \delta_k S_k\, e^{r(t_{k+1}-t_k)} \;+\; \delta_k\, S_{k+1} \\ &= e^{r(t_{k+1}-t_0)}\Big\{V_0 \;+\; \delta_0\big(S_1\, e^{-r(t_1-t_0)} - S_0\big) \qquad (1.22) \\ &\quad + \delta_1\big(S_2\, e^{-r(t_2-t_0)} - S_1\, e^{-r(t_1-t_0)}\big) \\ &\quad + \cdots \\ &\quad + \delta_{k-2}\big(S_{k-1}\, e^{-r(t_{k-1}-t_0)} - S_{k-2}\, e^{-r(t_{k-2}-t_0)}\big) \\ &\quad + \delta_{k-1}\big(S_k\, e^{-r(t_k-t_0)} - S_{k-1}\, e^{-r(t_{k-1}-t_0)}\big) \;-\; \delta_k S_k\, e^{-r(t_k-t_0)} \;+\; \delta_k\, S_{k+1}\, e^{-r(t_{k+1}-t_0)}\Big\} \end{aligned}$$

which is the same as

$$\begin{aligned} V_{k+1} &= e^{r(t_{k+1}-t_0)}V_0 + \sum_{j=1}^{k+1} \delta_{j-1}\left(e^{r(t_{k+1}-t_j)}S_j - e^{r(t_{k+1}-t_{j-1})}S_{j-1}\right) \\ &= e^{r(t_{k+1}-t_0)}\Big\{V_0 + \sum_{j=1}^{k+1} \delta_{j-1}\left(e^{-r(t_j-t_0)}S_j - e^{-r(t_{j-1}-t_0)}S_{j-1}\right)\Big\} \end{aligned}$$

This is formula (1.6) for $k+1$ and the induction step has been completed. ∎

Observe that at $k = N$, at maturity, we have

$$V_N = e^{r(t_N-t_0)}V_0 + \sum_{j=1}^{N} \delta_{j-1}\left(e^{r(t_N-t_j)}S_j - e^{r(t_N-t_{j-1})}S_{j-1}\right) \qquad (1.23)$$

and the right hand side of (1.23), in terms of $\delta$, is a function of

$$\delta_0,\ \delta_1,\ \cdots\ ,\ \delta_{N-2},\ \delta_{N-1} \qquad (1.24)$$

That is, there is no $\delta_N$. This is so because the stock position will be closed on day $t_N$, there is no preparation for a day $t_{N+1}$ to hold a number of $\delta_N$ stocks on that day, but all $\delta_{N-1}$ stocks which are hold on the beginning of day $t_N$ are simply sold on that day to finish with the trading strategy on that day.

**Replication of Option Payoffs**

Let

$$S_k = S(t_k) \tag{1.25}$$

again denote the price of some tradable asset. $S$ could be a stock like BASF or Deutsche Telekom, a stock index like the DAX, ESTOXX50, SPX or N225 or a commodity like oil, gold or silver or some foreign exchange (fx for short) rate like EUR/USD, GBP/USD or EUR/CHF.

**Definition 1.1:** An option with underlying $S$, start time $t_0$ and maturity $t_N$ is an arbitrary function

$$H : \{S_0, S_1, \cdots, S_{N-1}, S_N\} \rightarrow \mathbb{R} \tag{1.26}$$

and $H$ is called the option payoff.

Typically, an option is sold by a seller (for example a bank) on start date $t_0$ or a couple of days before start date to a buyer (a retail customer or a corporate client), and the option buyer receives the amount $H(S_0, S_1, ..., S_N)$ from the option seller at maturity date $t_N$. Apparently, at start time $t_0$ it is not clear how big or small the option payoff will be at maturity $t_N$ and thus it is not clear what "the" or a price for that option should be at start time. Maybe there is no well defined unique price and a price is simply determined by offer and demand. However, it is a fundamental result of financial mathematics that there is in fact a unique price and deviation from this price will result in arbitrage opportunities. Roughly speaking, one can make the following statement:

**Basic Principle of No-Arbitrage Pricing:** Let $S$ be some tradable asset and let $H$ be some option on $S$ with maturity $t_N$. Then, for a large class of realistic asset price models for $S$, the option payoff $H$ can be exactly replicated by a suitable trading strategy in the underlying (and by trading in plain vanilla options on $S$ (in case of stochastic volatility models) and by trading in bonds (in case of variable interest rates)). The unique cash amount $V_0$ which is needed to set up this replicating strategy is called the **theoretical fair value** (or just fair value) of the option.

Thus this course is mainly dealing with the following questions:

- come up with realistic asset price models
- construct the replicating trading strategies and calculate the fair value $V_0$