Chapter 8

Continuous Time Calculus and the Ito-Formula

In the last chapter we derived the Black-Scholes equation by considering the recursion relations of the replicating portfolio in the approximating Binomial model and then we took the continuous time limit. In this chapter we ask the following question: Is it possible to derive the Black-Scholes equation directly from the Black-Scholes model

$$dS_t/S_t \;=\; \mu\, dt + \sigma\, dx_t \tag{8.1}$$

in continuous time, without using the approximating Binomial model? The answer is yes, but in order to do so we need a mathematical tool called the Ito-formula which will be motivated alongside the following considerations. For simplicity, we start with zero rates, $r = 0$.

In the first chapter we saw that the portfolio value V_{t_k} of a selffinancing strategy, which holds $\delta_{t_{k-1}}$ stocks 'at the end of time t_{k-1}' or 'at the beginning of time t_k' and readjusts this to δ_{t_k} stocks at the end of time t_k after the asset price has switched from $S_{t_{k-1}}$ to S_{t_k}, is given by

$$V_{t_k} \;=\; V_0 + \sum_{j=1}^{k} \delta_{t_{j-1}} \cdot (S_{t_j} - S_{t_{j-1}}) \;=\; V_{t_{k-1}} + \delta_{t_{k-1}} \cdot (S_{t_k} - S_{t_{k-1}}) \tag{8.2}$$

In continuous time with 'continuous trading' (it is because of this notion that so far we have tried to derive everything up to now in discrete time, it is simply more intuitive) this may be rewritten as

$$V_t \;=\; V_0 + \textstyle\int_0^t \delta_\tau\, dS_\tau \tag{8.3}$$

or in differential form, if we subtract the $V_{t_{k-1}}$-term on the right hand side (8.2),

$$dV \;=\; \delta\, dS \tag{8.4}$$

where dV is the limit of

$$V_t(S_t) - V_{t-\Delta t}(S_{t-\Delta t}) \;=\; V(S_t,t) - V(S_{t-\Delta t}, t-\Delta t) \;\stackrel{\Delta t \to 0}{\to}\; dV \tag{8.5}$$

Differences like (8.5) are computed with Taylor expansion. One is tempted to write

$$dV = \frac{\partial V}{\partial S}\,dS + \frac{\partial V}{\partial t}\,dt \qquad \text{(not correct!)} \tag{8.6}$$

If this would be correct, then we should get similarly, for example, for the geometric Brownian motion $S_t = S_0\, e^{(\mu-\sigma^2/2)t+\sigma x_t}$ which is the solution to $dS/S = \mu\, dt + \sigma\, dx_t$ as we saw in chapter 4,

$$dS = \frac{\partial S}{\partial x}\,dx + \frac{\partial S}{\partial t}\,dt = \sigma S\,dx + \left(\mu - \frac{\sigma^2}{2}\right) S\,dt \qquad \text{(not correct!)} \tag{8.7}$$

which is apparently wrong. The reason behind this is that if one wants to compute differences $f(h(t+dt)) - f(h(t))$ where the function h has an infinite first variation (see below), which is the case for Brownian motion but which is not the case for almost all deterministic functions one encounters in standard calculus, then a first order Taylor expansion is not sufficient, but higher order terms have to be taken into account. If the quadratic variation of h is finite, which is the case for Brownian motion, then a second order Taylor expansion is sufficient. The corresponding formula is called the Ito Lemma which will be derived now. The result is summarized in Theorem 8.1 below.

Consider the composition of two functions $f \circ h\,(t) = f(h(t))$ where $h : [0,T] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. f is supposed to be a smooth function, at least two times differentiable, whereas for h we will make different assumptions. In the discussion above, $f = V$, the portfolio value, and $h(t) = S_t$, the price process.

Definition 8.1: Let $h : [0,T] \to \mathbb{R}$. The quantity

$$\mathrm{Var}_1(h) := \lim_{\varepsilon\to 0} \sup_{\substack{0=t_0<t_1<\cdots<t_N=T \\ |t_k-t_{k-1}|\le\varepsilon}} \sum_{k=1}^{N} |h(t_k) - h(t_{k-1})| \tag{8.8}$$

is called the first variation of h. Here the sup is taken over all not necessarily equally spaced decompositions $0 = t_0 < t_1 < \cdots < t_N = T$ where $|t_k - t_{k-1}| \le \varepsilon$ for all k. The quantity

$$\mathrm{Var}_2(h) := \lim_{\varepsilon\to 0} \sup_{\substack{0=t_0<t_1<\cdots<t_N=T \\ |t_k-t_{k-1}|\le\varepsilon}} \sum_{k=1}^{N} |h(t_k) - h(t_{k-1})|^2 \tag{8.9}$$

is called the quadratic variation of h.

Suppose h is differentiable. Then by the mean value theorem of calculus ($\xi_k \in [t_{k-1}, t_k]$)

$$\sum_{k=1}^{N} |h(t_k) - h(t_{k-1})| \;=\; \sum_{k=1}^{N} |h'(\xi_k)|(t_k - t_{k-1}) \overset{\varepsilon\to 0}{\to} \int_0^T |h'(t)|\,dt \tag{8.10}$$

since the sum is a Riemannian sum for the integral. Furthermore,

$$\begin{aligned}\sum_{k=1}^{N} |h(t_k) - h(t_{k-1})|^2 &\leq \sup_k\{|h(t_k) - h(t_{k-1})|\} \sum_{k=1}^{N} |h(t_k) - h(t_{k-1})| \\ &\leq \sup_{t\in[0,T]} \{|h'(t)|\}\, \varepsilon \sum_{k=1}^{N} |h(t_k) - h(t_{k-1})| \overset{\varepsilon\to 0}{\rightarrow} 0 \qquad (8.11)\end{aligned}$$

Thus, if h is a differentiable function,

$$h \in C^1([0,T]) \quad \Rightarrow \quad \begin{aligned} \mathrm{Var}_1(h) &= \int_0^T |h'(t)|\, dt \\ \mathrm{Var}_2(h) &= 0 \end{aligned} \qquad (8.12)$$

Now suppose $h(t) = x_t$ where $\{x_t\}_{0<t\leq T}$ is Brownian motion. In chapter 4 we saw that (see (4.27))

$$\lim_{\Delta t\to 0} \sum_{k=1}^{t/\Delta t} (x_{k\Delta t} - x_{(k-1)\Delta t})^2 = t \quad \text{with probability 1} \qquad (8.13)$$

That is, $\mathrm{Var}_2(x) = T$ or $\mathrm{Var}_2^{[0,t]}(x) = t$ with the obvious definition of $\mathrm{Var}_2^{[0,t]}(x)$. Furthermore, since the paths of Brownian motion are continuous,

$$\underbrace{\max_k\{|x_{k\Delta t} - x_{(k-1)\Delta t}|\}}_{\to 0 \text{ for } \Delta t\to 0} \sum_{k=1}^{t/\Delta t} |x_{k\Delta t} - x_{(k-1)\Delta t}| \geq \sum_{k=1}^{t/\Delta t} (x_{k\Delta t} - x_{(k-1)\Delta t})^2 \overset{\Delta t\to 0}{\rightarrow} t > 0 \qquad (8.14)$$

the sum on the left hand side of (8.14) has to diverge. Hence

$$x = \{x_t\}_{0<t\leq T} \text{ Brownian motion} \quad \Rightarrow \quad \begin{aligned} \mathrm{Var}_1^{[0,t]}(x) &= \infty \\ \mathrm{Var}_2^{[0,t]}(x) &= t \end{aligned} \qquad (8.15)$$

with probability one.

Now we consider the composition of two functions $f \circ h\,(t) = f(h(t))$ where f is smooth and h may be differentiable or may be a Brownian motion which is nowhere differentiable. For simplicity, we consider only equally spaced decompositions $t_k = k\Delta t$, $k = 0, 1, ..., N_T = T/\Delta t$ of the intervall $[0, T]$. Suppose first that h is differentiable. Then we can write ($N_t = t/\Delta t$)

$$
\begin{aligned}
f(h(t)) - f(h(0)) &= \sum_{k=1}^{N_t} \big(f(h(t_k)) - f(h(t_{k-1}))\big) \\
&= \sum_{k=1}^{N_t} f'(y_k)\big(h(t_k) - h(t_{k-1})\big)\,, \quad y_k \in [h(t_{k-1}), h(t_k)] \\
&\overset{\Delta t \to 0}{\to} \int_0^t f'(h)\, dh \qquad (8.16) \\
&= \int_0^t f'(h(s))\, h'(s)\, ds \qquad (8.17)
\end{aligned}
$$

Equation (8.17) holds only for differentiable h whereas equation (8.16) holds for any right continuous h which is of bounded first variation. In that case, the assignment $\mu_{1,h}((s,t]) := h(t) - h(s)$ defines a unique finite signed measure $\mu_{1,h}$ and the integral in (8.16) is the integral with respect to this measure, $dh \equiv d\mu_{1,h}$. Now assume that h has infinite first variation (then $\mu_{1,h}$ does not exist) but finite quadratic variation. Further let h be continuous. We Taylor expand f up to second order,

$$
f(y_1) - f(y_0) = f'(y_0)(y_1 - y_0) + \tfrac{1}{2} f''(y_0)(y_1 - y_0)^2 + O\big((y_1 - y_0)^3\big) \qquad (8.18)
$$

such that

$$
\begin{aligned}
f(h(t)) - f(h(0)) &= \sum_{k=1}^{N_t} \big(f(h(t_k)) - f(h(t_{k-1}))\big) \\
&= \sum_{k=1}^{N_t} f'(h(t_{k-1}))\big(h(t_k) - h(t_{k-1})\big) \qquad (8.19) \\
&\quad + \tfrac{1}{2} \sum_{k=1}^{N_t} f''(h(t_{k-1}))\big(h(t_k) - h(t_{k-1})\big)^2 \qquad (8.20) \\
&\quad + \sum_{k=1}^{N_t} O\big((h(t_k) - h(t_{k-1}))^3\big) \qquad (8.21)
\end{aligned}
$$

Because h has finite quadratic variation, the third order term (8.21) vanishes in the limit $\Delta t \to 0$, since

$$
\sum_{k=1}^{N_t} |h(t_k) - h(t_{k-1})|^3 \;\le\; \underbrace{\max_k \{|h(t_k) - h(t_{k-1})|\}}_{\to 0} \underbrace{\sum_{k=1}^{N_t} |h(t_k) - h(t_{k-1})|^2}_{\to \mathrm{Var}_2(h) < \infty} \;\to\; 0 \qquad (8.22)
$$

The limit of the first order term (8.19) (if it exists) we define as the integral

$$
\int_0^t f'(h(s))\, dh(s) := \lim_{\Delta t \to 0} \sum_{k=1}^{N_t} f'(h(t_{k-1}))\big(h(t_k) - h(t_{k-1})\big) \qquad (8.23)
$$

Consider the quadratic term (8.20). The assignment

$$(s,t] \to \mu_{2,h}\big((s,t]\big) := \lim_{\Delta t \to 0} \sum_{k=1}^{N_T} \chi_{(s,t]}(t_{k-1}) \left(h(t_k) - h(t_{k-1})\right)^2 \tag{8.24}$$

defines a unique measure $\mu_{2,h}$ on the Borel sets of $[0,T]$ and in the limit (8.20) becomes

$$\lim_{\Delta t \to 0} \sum_{k=1}^{N_t} f''(h(t_{k-1}))\big(h(t_k) - h(t_{k-1})\big)^2 \;=\; \int_0^t f''(h(s))\, d\mu_{2,h}(s) \tag{8.25}$$

A very intuitive notation is $d\mu_{2,h} \equiv (dh)^2$ which we will use from now on. If $h(t) = x_t$ is Brownian motion, then, by (8.13), $\mu_{2,x}\big((0,t]\big) = t$ and

$$d\mu_{2,x}(t) = (dx)^2 = dt \quad \text{with probability 1,} \tag{8.26}$$

compare also (8.46) below. We summarize our results in the following

Theorem 8.1 (Ito-Formula): Let $h : [0,T] \to \mathbb{R}$ be continuous and of bounded quadratic variation, $\mathrm{Var}_2(h) < \infty$. Let $f \in C^2(\mathbb{R})$. Then

$$f\big(h(t)\big) - f\big(h(0)\big) \;=\; \int_0^t f'\big(h(s)\big)\, dh(s) \;+\; \tfrac{1}{2} \int_0^t f''\big(h(s)\big)\, (dh(s))^2 \tag{8.27}$$

where the first integral is defined by

$$\int_0^t f'\big(h(s)\big)\, dh(s) := \lim_{\Delta t \to 0} \sum_{k=1}^{N} f'\big(h(t_{k-1})\big)\big(h(t_k) - h(t_{k-1})\big) \tag{8.28}$$

and the measure $(dh)^2 \equiv d\mu_{2,h}$ is defined by (8.24). In differential form, (8.27) is written as

$$d[f(h)] \;=\; f'(h)\, dh \;+\; \tfrac{1}{2} f''(h)\, (dh)^2 \tag{8.29}$$

The case $h(t) = x_t$ a Brownian motion where we have $(dx)^2 = dt$ is of particular importance and we put it into a separat

Corollary 8.1: Let $f \in C^2(\mathbb{R})$ and let x_t be a Brownian motion. Then

$$f(x_t) - f(x_0) \;=\; \int_0^t f'(x_s)\, dx_s \;+\; \tfrac{1}{2} \int_0^t f''(x_s)\, ds \tag{8.30}$$

where the first integral defined by

$$\int_0^t f'(x_s)\,dx_s := \lim_{\Delta t\to 0} \sum_{k=1}^{N} f'(x_{t_{k-1}})\big(x_{t_k} - x_{t_{k-1}}\big) \tag{8.31}$$

is called an Ito-integral. In differential form,

$$d\big(f(x_t)\big) \;=\; f'(x_t)\,dx_t \;+\; \tfrac{1}{2}f''(x_t)\,dt\;. \tag{8.32}$$

Let us illustrate (8.32) by applying it to the geometric Brownian motion given by $S_t = S_0\,e^{(\mu-\sigma^2/2)t+\sigma x_t}$. We write

$$S_t = f(x_t,t) \quad \text{where} \quad f(y,t) = S_0\,e^{\mu t+\sigma y-\frac{\sigma^2}{2}t} \tag{8.33}$$

Since in this case f depends also on t, there will be an additional term $\frac{\partial f}{\partial t}\,dt$ in (8.32). Thus

$$\begin{aligned} dS_t &= f'(x_t)\,dx_t + \tfrac{1}{2}f''(x_t)\,dt + \tfrac{\partial f}{\partial t}\,dt \\ &= \sigma\,S_t\,dx_t + \tfrac{\sigma^2}{2}\,S_t\,dt + \left(\mu - \tfrac{\sigma^2}{2}\right)\,S_t\,dt \\ &= \sigma\,S_t\,dx_t + \mu\,S_t\,dt \end{aligned} \tag{8.34}$$

which is exactly the SDE for the Black-Scholes model as it should be.

Now let us return to our original question, namely how can we derive the Black-Scholes equation directly in continuous time without making a reference to an approximating Binomial model. Let us first remark that, because of Excercise 8.2 below,

$$\begin{aligned} (dS)^2 &= (\sigma S\,dx_t + \mu\,S\,dt)^2 \\ &= \sigma^2 S^2 (dx_t)^2 \\ &= \sigma^2 S^2\,dt \end{aligned} \tag{8.35}$$

Now we are in a position to calculate dV, the change of the value of the replicating portfolio in contiuous time. With the Ito-Formula, we get

$$\begin{aligned} dV &= \tfrac{\partial V}{\partial S}\,dS + \tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\,(dS)^2 + \tfrac{\partial V}{\partial t}\,dt \\ &= \tfrac{\partial V}{\partial S}\,dS + \left(\tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \tfrac{\partial V}{\partial t}\right)\,dt \end{aligned} \tag{8.36}$$

Thus, if this change should be given by trading δ stocks of the underlying, that is, if this should be equal to $\delta\,dS$,

$$dV \;=\; \tfrac{\partial V}{\partial S}\,dS + \left(\tfrac{1}{2}\tfrac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \tfrac{\partial V}{\partial t}\right)\,dt \;\overset{!}{=}\; \delta\,dS \tag{8.37}$$

we have to have the equations

$$\delta = \frac{\partial V}{\partial S} \tag{8.38}$$

which coincides with (7.11) of the previous chapter and

$$\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial t} = 0 \tag{8.39}$$

which is the Black-Scholes equation for zero interest rates. Thus, if (8.38) and (8.39) are fulfilled, we get from Theorem 8.1 (or an extended version thereof to include the explicit time-dependency)

$$\begin{aligned} V(S_T,T) - V(S_0,0) &= \int_0^T \frac{\partial V}{\partial S}\, dS_t \;+\; \int_0^T \left(\frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{\partial V}{\partial t}\right) dt \\ &= \int_0^T \delta(S_t,t)\, dS_t \end{aligned} \tag{8.40}$$

and some payoff $H = H(S_T)$ will be exactly replicated in continuous time if we impose the final condition $V(S_T,T) = H(S_T)$ in addition to (8.39).

When interest rates are present, a similar derivation can be made. This will be done in the exercises.

Ito-Integral and Stratonovich-Integral

Finally we have to discuss a subtlety concerning the definition of the integral in (8.28),

$$\int_0^t f'\big(h(s)\big)\, dh(s) := \lim_{\Delta t \to 0} \sum_{k=1}^{N} f'\big(h(t_{k-1})\big)\big(h(t_k) - h(t_{k-1})\big) \tag{8.41}$$

The function f is nice, say, C^2, but the function h has infinite first variation and bounded quadratic variation. The point is that the definition (8.41) depends on the place where f' on the right hand side of (8.41) is evaluated; we get a different result if we write, for example, $f'(\bar{y}_k)$ where $\bar{y}_k = \big(h(t_{k-1}) + h(t_k)\big)/2$. To see this more clearly recall (8.18-8.21). Suppose that, in (8.18), we Taylor expand around $\bar{y} = (y_0 + y_1)/2$ instead of y_0. Then

$$\begin{aligned} f(y_1) - f(y_0) &= f(y_1) - f(\bar{y}) + f(\bar{y}) - f(y_0) \\ &= f'(\bar{y})(y_1 - \bar{y}) + \tfrac{1}{2} f''(\bar{y})(y_1 - \bar{y})^2 \\ &\qquad + f'(\bar{y})(\bar{y} - y_0) - \tfrac{1}{2} f''(\bar{y})(\bar{y} - y_0)^2 + O\left((y_1 - y_0)^3\right) \\ &= f'(\bar{y})(y_1 - y_0) + \tfrac{1}{2} f''(\bar{y}) \underbrace{\left((y_1 - \bar{y})^2 - (\bar{y} - y_0)^2\right)}_{=0} + O\left((y_1 - y_0)^3\right) \end{aligned} \tag{8.42}$$

Hence, with the definition

$$\int_0^t f'\big(h(s)\big) \circ dh(s) := \lim_{\Delta t \to 0} \sum_{k=1}^{N} f'(\bar{y}_k)\big(h(t_k) - h(t_{k-1})\big) \tag{8.43}$$

$\bar{y}_k := (h(t_{k-1}) + h(t_k))/2$, the second term in the Ito-formula (8.27) would be completely absent and we have

$$\int_0^t f'\big(h(s)\big) \circ dh(s) = \int_0^t f'\big(h(s)\big)\, dh(s) + \tfrac{1}{2} \int_0^t f''\big(h(s)\big)\, (dh(s))^2 \tag{8.44}$$

In the book of Oksendal [12](8.41) is called an Ito integral and (8.43) is called a Stratonovich integral. It is very instructive to confirm (8.44) with an Excel simulation.

For our purposes it is apparently exactly the Ito-integral which is needed. Namely, we need these integrals to represent the values of trading strategies and these values are given by, for zero rates, (8.3)

$$V_t \;=\; V_0 + \int_0^t \delta_\tau \, dS_\tau$$

which we have to define as the continuous time limit of (8.2),

$$V_{t_k} \;=\; V_0 + \sum_{j=1}^{k} \delta_{t_{j-1}} \cdot (S_{t_j} - S_{t_{j-1}})$$

and in the equation above it is absolutely crucial that the δ can depend only on quantities which are known at time t_{j-1}, since the readjustment is done 'at the end of time t_{j-1}' where the stock price is still $S_{t_{j-1}}$.

Excercise 8.1: Let $f : [0,T] \to \mathbb{R}$ be continuous and let $t_k = k\Delta t$, $k = 0, 1, ..., N_T = T/\Delta t$. Let

$$I_{\Delta t}(f) := \sum_{k=1}^{N_T} f(t_{k-1}) \left(x_{t_k} - x_{t_{k-1}}\right)^2 \tag{8.45}$$

Show that

$$\lim_{\Delta t \to 0} \mathsf{P}\Big(\big|I_{\Delta t}(f) - \int_0^T f(t)\, dt\big| \geq \varepsilon\Big) \;=\; 0 \tag{8.46}$$

for any $\varepsilon > 0$. To this end, prove the following items:

a) Prove Chebyshevs inequality,

$$\mathsf{P}\big(|X - \mathsf{E}[X]| \geq \varepsilon\big) \;\leq\; \tfrac{1}{\varepsilon^2}\, \mathsf{V}[X] \tag{8.47}$$

for any random variable X.

b) Show that

$$\lim_{\Delta t \to 0} \mathsf{E}[I_{\Delta t}(f)] = \int_0^T f(t)\, dt\,, \qquad \lim_{\Delta t \to 0} \mathsf{V}[I_{\Delta t}(f)] = 0 \tag{8.48}$$

Excercise 8.2: Let $f, g : [0,T] \to \mathbb{R}$ be continuous where f has finite quadratic variation and g has finite first variation. Prove that

$$\mathrm{Var}_2(f+g) = \mathrm{Var}_2(f) \tag{8.49}$$

In particular, $\mu_{2,f+g} = \mu_{2,f}$ or

$$\big(d(f+g)\big)^2 = (df)^2 \quad \text{if } \mathrm{Var}_1(g) < \infty \tag{8.50}$$

Solutions to Exercises

Excercise 8.1: a) We have

$$\begin{aligned} \mathsf{V}[X] &= \textstyle\int (X - \mathsf{E}[X])^2 dP \\ &= \textstyle\int_{|X-\mathsf{E}[X]|<\varepsilon} (X - \mathsf{E}[X])^2 dP + \int_{|X-\mathsf{E}[X]|\geq\varepsilon} (X - \mathsf{E}[X])^2 dP \\ &\geq \textstyle\int_{|X-\mathsf{E}[X]|\geq\varepsilon} (X - \mathsf{E}[X])^2 dP \\ &\geq \textstyle\int_{|X-\mathsf{E}[X]|\geq\varepsilon} \varepsilon^2 dP \;=\; \varepsilon^2 \mathsf{P}\big(|X - \mathsf{E}[X]| \geq \varepsilon\big) \end{aligned} \tag{8.51}$$

which proves part (a).

b) Because of $\mathsf{E}[(x_s - x_t)^2] = |s-t|$, compare (**??**), we have

$$\begin{aligned} \mathsf{E}[I_{\Delta t}(f)] &= \sum_{k=1}^{N_T} f(t_{k-1})\, \mathsf{E}\big[(x_{t_k} - x_{t_{k-1}})^2\big] \\ &= \sum_{k=1}^{N_T} f(t_{k-1})\,(t_k - t_{k-1}) \\ &\overset{\Delta t \to 0}{\to} \int_0^T f(t)\,dt \end{aligned} \tag{8.52}$$

Since for, say, $s < t$,

$$\begin{aligned} \mathsf{V}\big[(x_t - x_s)^2\big] &= \mathsf{E}\big[(x_t - x_s)^4\big] - \mathsf{E}\big[(x_t - x_s)^2\big]^2 \\ &= \tfrac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} (x_t - x_s)^4\, e^{-\frac{(x_t - x_s)^2}{2(t-s)}}\, d(x_t - x_s) \;-\; (t-s)^2 \\ &= \tfrac{1}{\sqrt{2\pi}} (t-s)^2 \int_{\mathbb{R}} y^4\, e^{-\frac{y^2}{2}}\, dy \;-\; (t-s)^2 \\ &= 3(t-s)^2 - (t-s)^2 = 2(t-s)^2 \end{aligned} \tag{8.53}$$

the variance of I_Δ is given by

$$\begin{aligned}
\mathsf{V}[I_{\Delta t}(f)] &= \sum_{k=1}^{N_T} f(t_{k-1})^2\,\mathsf{V}\big[(x_{t_k}-x_{t_{k-1}})^2\big] \\
&= 2\sum_{k=1}^{N_T} f(t_{k-1})^2\,\Delta t^2 \\
&= 2\Delta t\,\underbrace{\sum_{k=1}^{N_T} f(t_{k-1})^2\,\Delta t}_{\to\int_0^T f(t)^2dt} \\
&\to 0 \quad \text{as} \quad \Delta t \to 0
\end{aligned} \tag{8.54}$$

This proves part (b).

Excercise 8.2: The quadratic variation is given by

$$\mathrm{Var}_2(f) = \lim_{\varepsilon\to 0} \sup_{\substack{0=t_0<t_1<\cdots<t_N=T \\ |t_k-t_{k-1}|\le\varepsilon}} \sum_{k=1}^{N} |f(t_k)-f(t_{k-1})|^2 \tag{8.55}$$

For fixed ε,

$$\begin{aligned}
&\sum_{k=1}^{N}\big|(f+g)(t_k)-(f+g)(t_{k-1})\big|^2 \qquad (8.56)\\
&= \sum_{k=1}^{N}\Big([f(t_k)-f(t_{k-1})]^2 + 2[f(t_k)-f(t_{k-1})][g(t_k)-g(t_{k-1})] + [g(t_k)-g(t_{k-1})]^2\Big)
\end{aligned}$$

and the second and last term in (8.56) vanishes for $\varepsilon \to 0$ because

$$\sum_{k=1}^{N}\big|[f(t_k)-f(t_{k-1})][g(t_k)-g(t_{k-1})]\big| \;\le\; \left\{\sum_{k=1}^{N}[f(t_k)-f(t_{k-1})]^2\sum_{j=1}^{N}[g(t_j)-g(t_{j-1})]^2\right\}^{\frac{1}{2}}$$

and $\mathrm{Var}_2(g)=0$.