

Chapter 9

The Risk Neutral Pricing Measure for the Black-Scholes Model

The discounted portfolio value of a selffinancing strategy in discrete time was given by

$$v_{t_k} = v_0 + \sum_{j=1}^k \delta_{t_{j-1}} (s_{t_j} - s_{t_{j-1}}) \quad (9.1)$$

where $v_{t_k} = e^{-rt_k} V_{t_k}$ is the discounted portfolio value at time $t_k = k\Delta t$, $k = 0, 1, \dots, N_T = T/\Delta t$ and $s_{t_k} = e^{-rt_k} S_{t_k}$ denotes the discounted asset price. Suppose that the price process S_{t_k} is given by a geometric Brownian motion such that

$$s_{t_k} = e^{-rt_k} S_0 e^{(\mu - \frac{\sigma^2}{2})t_k + \sigma x_{t_k}} = S_0 e^{\sigma x_{t_k} + (\mu - r - \sigma^2/2)t_k} =: s(x_{t_k}, t_k) \quad (9.2)$$

Suppose further that we have some european option with payoff H or discounted payoff $h = e^{-rT} H$ where $h = h(S_{t_N})$ or more generally $h = h(S_{t_0}, \dots, S_{t_N})$, $N = N_T$, and suppose that there is a replicating strategy

$$\delta_{t_k} = \delta_{t_k}(S_{t_1}, \dots, S_{t_k}) \quad (9.3)$$

such that

$$h = v_{t_N} = v_0 + \sum_{k=1}^N \delta_{t_{k-1}} (s_{t_k} - s_{t_{k-1}}) \quad (9.4)$$

Since $v_0 = V_0$, the price of the option, is a number, we can write

$$v_0 = \mathbb{E}[v_0] = \mathbb{E}[h] - \sum_{k=1}^N \mathbb{E}[\delta_{t_{k-1}} (s_{t_k} - s_{t_{k-1}})] \quad (9.5)$$

where the expectation in (9.5) can be chosen arbitrarily. Let us consider (9.5) for the Wiener measure,

$$dW(\{x_t\}_{0 < t \leq T}) = \lim_{\Delta t \rightarrow 0} \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t}) dx_{k\Delta t} \quad (9.6)$$

Because of Theorem 4.1 we have

$$\begin{aligned}
& \mathbb{E}_W[\delta_{k-1}(s_k - s_{k-1})] \tag{9.7} \\
&= \int_{\mathbb{R}^k} \delta_{k-1}(x_{t_1}, \dots, x_{t_{k-1}}) (s(x_{t_k}, t_k) - s(x_{t_{k-1}}, t_{k-1})) \prod_{j=1}^k p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j}) dx_{t_j} \\
&= \int_{\mathbb{R}^{k-1}} \delta_{k-1}(x_{t_1}, \dots, x_{t_{k-1}}) \left(\int_{\mathbb{R}} s(x_{t_k}, t_k) p_{t_k - t_{k-1}}(x_{t_{k-1}}, x_{t_k}) dx_{t_k} - s(x_{t_{k-1}}, t_{k-1}) \right) \times \\
&\quad \prod_{j=1}^{k-1} p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j}) dx_{t_j} \\
&= \int_{\mathbb{R}^{k-1}} \delta_{k-1}(x_{t_1}, \dots, x_{t_{k-1}}) \left(\mathbb{E}_W[s(x_{t_k}, t_k) | x_{t_{k-1}}] - s(x_{t_{k-1}}, t_{k-1}) \right) \prod_{j=1}^{k-1} p_{t_j - t_{j-1}}(x_{t_{j-1}}, x_{t_j}) dx_{t_j}
\end{aligned}$$

where we introduced the conditional expectation

$$\mathbb{E}_W[f(\{x_s\}_{0 \leq s \leq T}) | x_t] := \int f(\{x_s\}) dW(\{x_s\}_{t < s \leq T}) \tag{9.8}$$

with the obvious definition ($t = N_t \Delta t$)

$$dW(\{x_t\}_{t < s \leq T}) := \lim_{\Delta t \rightarrow 0} \prod_{j=N_t+1}^{N_T} p_{\Delta t}(x_{(j-1)\Delta t}, x_{j\Delta t}) dx_{j\Delta t} \tag{9.9}$$

In particular, if the function f in (9.8) depends only on a single x_s , $f = f(x_s)$, then

$$\mathbb{E}_W[f(x_s) | x_t] = \begin{cases} f(x_s) & \text{if } t \geq s \\ \int_{\mathbb{R}} f(x_s) p_{s-t}(x_t, x_s) dx_s & \text{if } t < s \end{cases} \tag{9.10}$$

Let us compute the conditional expectation $\mathbb{E}_W[s(x_{t_k}, t_k) | x_{t_{k-1}}]$ in the last line of (9.7). We have, using $\Delta t = t_k - t_{k-1}$

$$\begin{aligned}
\mathbb{E}_W[s(x_{t_k}, t_k) | x_{t_{k-1}}] &= \int_{\mathbb{R}} s(x_{t_k}, t_k) p_{t_k - t_{k-1}}(x_{t_{k-1}}, x_{t_k}) dx_{t_k} \\
&= \frac{1}{\sqrt{2\pi\Delta t}} \int_{\mathbb{R}} S_0 e^{\sigma x_{t_k} + (\mu - r - \sigma^2/2)t_k} e^{-\frac{(x_{t_k} - x_{t_{k-1}})^2}{2\Delta t}} dx_{t_k} \\
&= S_0 e^{\sigma x_{t_{k-1}} + (\mu - r - \sigma^2/2)t_k} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\sigma\sqrt{\Delta t}y} e^{-\frac{y^2}{2}} dy \\
&= S_0 e^{\sigma x_{t_{k-1}} + (\mu - r - \sigma^2/2)t_k} e^{\frac{\sigma^2}{2}(t_k - t_{k-1})} \\
&= s(x_{t_{k-1}}, t_{k-1}) e^{(\mu - r)(t_k - t_{k-1})} \tag{9.11}
\end{aligned}$$

Now, suppose the factor $e^{(\mu - r)(t_k - t_{k-1})}$ in (9.11) would be absent. Then the round brackets in the third line of (9.7) would be zero for all k and the price of the option v_0 would be

given by the expectation of the discounted payoff. Thus, we would be able to compute the price without knowing the hedging strategy, provided that there is a replicating strategy. Now, this factor is not absent but we can ask the following question: Is there some measure $d\tilde{W}$ such that

$$\mathbf{E}_{\tilde{W}}[s(x_{t_k}, t_k) | x_{t_{k-1}}] = s(x_{t_{k-1}}, t_{k-1}) \quad (9.12)$$

If this is the case then we can compute the price v_0 of the option with discounted payoff h by taking the expectation value with respect to $d\tilde{W}$,

$$v_0 = \mathbf{E}_{\tilde{W}}[h] \quad (9.13)$$

since the round brackets in the third line of (9.7) all vanish. There is the following

Theorem 9.1: Let $s_{t_k} = s(x_{t_k}, t_k)$ be a discounted geometric Brownian motion given by (9.2). Define the kernels $\tilde{p}_t(x, y) = \tilde{p}_t^{\mu, r, \sigma}(x, y)$ by

$$\tilde{p}_t(x, y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y-\frac{\mu-r}{\sigma}t)^2}{2t}} \quad (9.14)$$

Then:

a) The kernels \tilde{p} satisfy

$$\int_{\mathbb{R}} \tilde{p}_s(x, y) \tilde{p}_t(y, z) dy = \tilde{p}_{s+t}(x, z) \quad (9.15)$$

and $\int_{\mathbb{R}} \tilde{p}_s(x, y) dy = 1$ for all x . That is, the measure

$$d\tilde{W}(\{x_t\}_{0 < t \leq T}) := \lim_{\Delta t \rightarrow 0} \prod_{k=1}^{N_T} \tilde{p}_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t}) dx_{k\Delta t} \quad (9.16)$$

is well defined.

b) The price process (9.2) is a martingale with respect to $d\tilde{W}$. That is,

$$\begin{aligned} \mathbf{E}_{\tilde{W}}[s(x_{t'}, t') | x_t] &:= \int s(x_{t'}, t') d\tilde{W}(\{x_s\}_{t < s \leq T}) \\ &= s(x_t, t) \quad \forall t < t' \end{aligned} \quad (9.17)$$

The measure $d\tilde{W}$ is called an equivalent martingale measure (with respect to the price process s_t).

Proof: a) Let $p_t(x, y)$ be the kernel (4.13). Then

$$\tilde{p}_t(x, y) = p_t\left(x - \frac{\mu-r}{\sigma}t, y\right) = p_t\left(x, y + \frac{\mu-r}{\sigma}t\right) = p_t\left(x - \frac{\mu-r}{2\sigma}, y + \frac{\mu-r}{2\sigma}\right) \quad (9.18)$$

such that, with Lemma 4.1,

$$\begin{aligned} \int_{\mathbb{R}} \tilde{p}_s(x, y) \tilde{p}_t(y, z) dy &= \int_{\mathbb{R}} p_s\left(x - \frac{\mu-r}{\sigma}s, y\right) p_t\left(y, z + \frac{\mu-r}{\sigma}t\right) dy \\ &= p_{s+t}\left(x - \frac{\mu-r}{\sigma}s, z + \frac{\mu-r}{\sigma}t\right) = \tilde{p}_{s+t}(x, z) \end{aligned} \quad (9.19)$$

which proves part (a). Part (b) is obtained as (9.11),

$$\begin{aligned} \mathbb{E}_{\tilde{W}}[s(x_{t'}, t') | x_t] &= \int_{\mathbb{R}} s(x_{t'}, t') \tilde{p}_{t'-t}(x_t, x_{t'}) dx_{t'} \\ &= \frac{1}{\sqrt{2\pi(t'-t)}} \int_{\mathbb{R}} S_0 e^{\sigma x_{t'} + (\mu-r-\sigma^2/2)t'} e^{-\frac{(x_t - x_{t'} - \frac{\mu-r}{\sigma}(t'-t))^2}{2(t'-t)}} dx_{t'} \\ &= S_0 e^{\sigma x_t + (\mu-r-\sigma^2/2)t} e^{-\frac{\sigma^2}{2}(t'-t)} \frac{1}{\sqrt{2\pi(t'-t)}} \int_{\mathbb{R}} e^{\sigma(x_{t'} - x_t + \frac{\mu-r}{\sigma}(t'-t))} e^{-\frac{(x_t - x_{t'} - \frac{\mu-r}{\sigma}(t'-t))^2}{2(t'-t)}} dx_{t'} \\ &= S_0 e^{\sigma x_t + (\mu-r-\sigma^2/2)t} e^{-\frac{\sigma^2}{2}(t'-t)} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\sigma\sqrt{t'-t}y} e^{-\frac{y^2}{2}} dy \\ &= S_0 e^{\sigma x_t + (\mu-r-\sigma^2/2)t} \\ &= s(x_t, t) \end{aligned} \quad (9.20)$$

This proves the theorem. ■

In chapter 5 where we approximated the Black-Scholes model with a suitable Binomial model, we were able to prove the following pricing formula for some non path dependent option with payoff $H = H(S_T)$, see Theorem 5.2:

$$V_0^{\text{BS}} = e^{-rT} \int_{\mathbb{R}} H(S_0 e^{(r-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x}) e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \quad (9.21)$$

Let us rederive (9.21) by using the equivalent martingale measure. Suppose we have a european option with discounted payoff $h(S_T) = e^{-rT}H(S_T)$ and that the dynamics of S is given by the Black-Scholes model with drift μ and volatility σ ,

$$S_T = S_0 e^{\sigma x_T + (\mu - \frac{\sigma^2}{2})T} \quad (9.22)$$

Observe that the drift parameter μ does not show up in the pricing formula (9.21). This was actually a quite fundamental result of chapter 5. Here we will come up with the same conclusion:

Let δ be the replicating strategy. Then

$$v_T = h(S_T) = v_0 + \sum_{k=1}^{N_T} \delta_{t_{k-1}} (s_{t_k} - s_{t_{k-1}}) \quad (9.23)$$

We fix some $t = t_k$ and take the expectation with respect to $d\tilde{W}(\{x_s\}_{t < s \leq T})$. This gives

$$\mathbf{E}_{\tilde{W}}[h(S_T)|x_t] = v_0 + \sum_{j=1}^k \delta_{t_{j-1}} (s_{t_j} - s_{t_{j-1}}) = v_{t_k} = v_t = e^{-rt} V_t \quad (9.24)$$

and we get

$$\begin{aligned} V_t &= e^{rt} \mathbf{E}_{\tilde{W}}[h(S_T)|x_t] \\ &= e^{-r(T-t)} \int_{\mathbb{R}} H(S_0 e^{\sigma x_T + (\mu - \frac{\sigma^2}{2})T}) \tilde{p}_{T-t}(x_t, x_T) dx_T \\ &= e^{-r(T-t)} \int_{\mathbb{R}} H(S_0 e^{\sigma x_T + (\mu - \frac{\sigma^2}{2})T}) e^{-\frac{(x_t - x_T - \frac{\mu-r}{\sigma}(T-t))^2}{2(T-t)}} \frac{dx_T}{\sqrt{2\pi(T-t)}} \\ &= e^{-r(T-t)} \int_{\mathbb{R}} H(S_t e^{\sigma(x_T - x_t) + (\mu-r)(T-t)} e^{-\frac{\sigma^2}{2}(T-t)} e^{r(T-t)}) e^{-\frac{(x_T - x_t + \frac{\mu-r}{\sigma}(T-t))^2}{2(T-t)}} \frac{dx_T}{\sqrt{2\pi(T-t)}} \\ &= e^{-r(T-t)} \int_{\mathbb{R}} H(S_t e^{\sigma y + (r - \frac{\sigma^2}{2})(T-t)}) e^{-\frac{y^2}{2(T-t)}} \frac{dy}{\sqrt{2\pi(T-t)}} \\ &= e^{-r(T-t)} \int_{\mathbb{R}} H(S_t e^{\sigma\sqrt{T-t}y + (r - \frac{\sigma^2}{2})(T-t)}) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \end{aligned} \quad (9.25)$$

and, for $t = 0$, this coincides with (9.21). More generally, there holds the following

Theorem 9.2: Let $\{x_t\}_{0 < t \leq T}$ be a Brownian motion and let

$$S_t^{(\mu)} := S_0 e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t} \quad (9.26)$$

Let dW be the Wiener measure and $d\tilde{W}$ be the equivalent martingale measure. Then the following equality

$$\mathbf{E}_{\tilde{W}}[H(S_{t_1}^{(\mu)}, \dots, S_{t_m}^{(\mu)})] = \mathbf{E}_W[H(S_{t_1}^{(r)}, \dots, S_{t_m}^{(r)})] \quad (9.27)$$

holds for all payoffs $H = H(S_{t_1}, \dots, S_{t_m})$. In particular, the theoretical fair value

$$\begin{aligned} V_0 &= e^{-rT} \mathbf{E}_{\tilde{W}}[H(S_{t_1}^{(\mu)}, \dots, S_{t_m}^{(\mu)})] \\ &= e^{-rT} \mathbf{E}_W[H(S_{t_1}^{(r)}, \dots, S_{t_m}^{(r)})] \end{aligned} \quad (9.28)$$

does not depend on the (usually not predictable) drift parameter μ , but depends only on the volatility parameter σ and the interest rate level r .

Proof: From the definition of \tilde{p} we have ($t_0 := 0$)

$$\mathbb{E}_{\tilde{W}}[H(S_{t_1}^{(\mu)}, \dots, S_{t_m}^{(\mu)})] = \int_{\mathbb{R}^m} H(S_{t_1}^{(\mu)}, \dots, S_{t_m}^{(\mu)}) \prod_{j=1}^m \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} e^{-\frac{[x_{t_{j-1}} + \frac{\mu-r}{\sigma}t_{j-1} - (x_{t_j} + \frac{\mu-r}{\sigma}t_j)]^2}{2(t_j - t_{j-1})}} dx_{t_j}$$

Because of

$$\begin{aligned} S_t^{(\mu)} &= S_t^{(\mu)}(x_t) = S_0 e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t} \\ &= S_0 e^{\sigma(x_t + \frac{\mu-r}{\sigma}t) + (r - \frac{\sigma^2}{2})t} \\ &= S_t^{(r)}(x_t + \frac{\mu-r}{\sigma}t) \end{aligned} \tag{9.29}$$

the statement follows from the substitution of variables $y_{t_j} = x_{t_j} + \frac{\mu-r}{\sigma}t_j$. ■