Chapter 3

Real World and Risk Neutral Probabilities and the Definition of Martingale

As we have remarked after Definition 2.2, we have not specified any probabilities in the definition of the Binomial model. We did that since, from the view point of option pricing, the decisive property of this model is that, in going from one time step to the next, there are only two possible choices. As a consequence, we could prove in Theorem 2.1 above that in this model every option payoff $H = H(S_0, S_1, \cdots, S_N)$ can be replicated exactly by a suitable trading strategy in the underlying S. Furthermore, we were able write down some recursion relations which put us in a position to calculate the replicating strategy $\{\delta_k\}_{k=0,1,\ldots,N-1}$ with $\delta_k = \delta_k(S_0, .., S_k)$ and the option price V_0.

In remark (1) following Theorem 2.1 we pointed out that if the option payoff is not path dependent but depends only on the underlying price at maturity, $H = H(S_N)$, then also the replicating strategy δ_k and the portfolio values V_k at time step t_k are not path dependent, but depend only on the underlying price $S_k = S(t_k)$. As a consequence, to calculate the option price of some non path dependent option $H = H(S_N)$, it is sufficient to consider the following tree structure, a recombining binomial tree with $n+1$ leaves at time step t_n, and to assign portfolio values to each node of this tree:

n-period recombining binomial tree with $n+1$ leaves, $n = 3$

However, if we want to price some path dependent option, then, if we want to do this with Theorem 2.1 by using the recursion relations (2.14), we have to consider the following non recombining binary tree structure with 2^n leaves at time step t_n:

n-period binary tree with 2^n leaves, $n = 3$

Since for example $2^{250} \approx 10^{75.26}$ is not that much different (well, by a factor of 100'000) from the number of estimated atoms in the universe, 10^{80}, it is obvious that using (2.14) is not practical for the pricing of path dependent (or so called 'exotic') derivatives. Thus, a different method is needed and fortunately there is a much more practical method.

Risk Neutral Probabilities

We go back to Definition 2.2 and introduce some probabilities. That is, we write

$$S_k \;=\; S_{k-1} \times \begin{cases} (1+\mathrm{ret}_{\mathrm{up}}) & \text{with some probability } \; p \\ (1+\mathrm{ret}_{\mathrm{down}}) & \text{with probability } \; 1-p \end{cases} \tag{3.1}$$

thereby making the price process $\{S_k\}_{k=0}^N$ to a stochastic process. We know already that option prices do not depend on p. Now we use that fact to make a special choice for p which will allow us to calculate option prices, also for path dependent options, in a practical and efficient way. From Theorem 2.1 we know that payoff replication is possible. For zero interest rates, we have (the general case $r > 0$ will be considered below)

$$H(S_0, S_1, ..., S_N) \;=\; V_0 + \sum_{k=1}^{N} \delta_{k-1}(S_0, ..., S_{k-1}) \times (S_k - S_{k-1}) \tag{3.2}$$

and V_0, the money which is needed to set up the replicating strategy, is the option price, the theoretical fair value of H. Let us introduce the following notation: For any function $f = f(S_0, S_1, ..., S_N)$ of the price process $\{S_k\}_{k=0}^N$ we introduce, for time t_k, a so called conditional expectation

$$\mathsf{E}\big[\, f(S_0, S_1, ..., S_N) \,|\, \{S_j\}_{j=0}^k \,\big] \tag{3.3}$$

by considering the time point t_k as the actual present time such that $S_k, S_{k-1}, \ldots, S_0$ are actually known but the prices $S_{k+1}, S_{k+2}, ..., S_N$ are unknown since they are still in the future. That is, the $S_0, ..., S_k$ are deterministic quantities given by some realization of returns $\mathrm{ret}_{\mathrm{up}}$, $\mathrm{ret}_{\mathrm{down}}$, $\mathrm{ret}_{\mathrm{down}}$, ... , $\mathrm{ret}_{\mathrm{up}}$ (k returns have realized), but the $S_{k+1}, S_{k+2}, ..., S_N$ are still random, stochastic quantities since the future returns haven't realized yet. To illustrate the concept, let us calculate the conditional expectation

$$\mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \tag{3.4}$$

According to (3.1) we have

$$S_{k+1} \;=\; S_k \times (1 + \mathrm{ret}_{k+1}) \tag{3.5}$$

with

$$\mathrm{ret}_{k+1} \;=\; \begin{cases} \mathrm{ret}_{\mathrm{up}} & \text{with probability } \; p \\ \mathrm{ret}_{\mathrm{down}} & \text{with probability } \; 1-p \end{cases} \tag{3.6}$$

Thus,

$$\begin{aligned} \mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;&=\; \mathsf{E}\big[\, S_k \times (1 + \mathrm{ret}_{k+1}) \,|\, \{S_j\}_{j=0}^k \,\big] \\ &=\; S_k \times \mathsf{E}\big[\, 1 + \mathrm{ret}_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \\ &=\; S_k \times \Big(1 \,+\, \mathsf{E}\big[\, \mathrm{ret}_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big]\Big) \\ &=\; S_k \times \Big(1 \,+\, \mathrm{ret}_{\mathrm{up}} \cdot p \,+\, \mathrm{ret}_{\mathrm{down}} \cdot (1-p)\Big) \end{aligned} \tag{3.7}$$

The choice of p which makes the conditional expectation (3.7) equal to S_k

$$\mathsf{E}\big[\, S_{k+1} \,|\, \{S_j\}_{j=0}^{k} \,\big] \;\stackrel{!}{=}\; S_k \tag{3.8}$$

is called the risk neutral probability (in case of zero interest rates). A stochastic process which fulfills equation (3.8) for all k is called a martingale. For zero interest rates, this risk neutral probability is obtained through

$$\begin{aligned} S_k \times \big(1 + \mathrm{ret_{up}}p + \mathrm{ret_{down}}(1-p)\big) &\stackrel{!}{=} S_k \\ \Leftrightarrow \quad \mathrm{ret_{up}}p + \mathrm{ret_{down}}(1-p) &= 0 \\ \Leftrightarrow \quad (\mathrm{ret_{up}} - \mathrm{ret_{down}})p &= -\mathrm{ret_{down}} \end{aligned}$$

which gives

$$p \;=\; \frac{-\mathrm{ret_{down}}}{\mathrm{ret_{up}} - \mathrm{ret_{down}}} \;=:\; p_{\text{risk neutral}} \tag{3.9}$$

Apparently the down return $\mathrm{ret_{down}}$ has to be a negative number to obtain a meaningful p. Now let us fix p to this value (3.9) and to be more explicit we will use the notation $\mathsf{E} = \mathsf{E}_{\mathrm{rn}}$, 'rn' for 'risk neutral', to indicate that we are calculating expectation values using the risk neutral probability (3.9). The importance of this definition is due to the following important calculation:

$$\begin{aligned} V_0 &= \mathsf{E}_{\mathrm{rn}}[\,V_0\,] \;=\; \mathsf{E}_{\mathrm{rn}}[\,V_0 \,|\, S_0\,] \\ &= \mathsf{E}_{\mathrm{rn}}\Big[\, H(S_0,S_1,...,S_N) \;-\; \sum_{k=1}^{N} \delta_{k-1}(S_0,...,S_{k-1}) \times (S_k - S_{k-1}) \,\Big|\, S_0 \,\Big] \\ &= \mathsf{E}_{\mathrm{rn}}\big[\, H(S_0,S_1,...,S_N)\,\big] \;-\; \sum_{k=1}^{N} \mathsf{E}_{\mathrm{rn}}\Big[\, \delta_{k-1}(S_0,...,S_{k-1}) \times (S_k - S_{k-1}) \,\Big|\, S_0 \Big] \end{aligned} \tag{3.10}$$

The expectations in the sum on the right hand side of (3.10) can be calculated as follows:

$$\begin{aligned} &\mathsf{E}_{\mathrm{rn}}\Big[\, \delta_{k-1}(S_0,...,S_{k-1}) \times (S_k - S_{k-1}) \,\Big|\, S_0 \Big] \;= \\ &= \mathsf{E}_{\mathrm{rn}}\Big[\, \underbrace{\mathsf{E}_{\mathrm{rn}}\Big[\, \delta_{k-1}(S_0,...,S_{k-1}) \times (S_k - S_{k-1}) \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big]}_{\text{in this expectation all } S_1,...,S_{k-1} \text{ are constant}} \,\Big|\, S_0 \,\Big] \\ &= \mathsf{E}_{\mathrm{rn}}\Big[\, \delta_{k-1}(S_0,...,S_{k-1}) \times \mathsf{E}_{\mathrm{rn}}\Big[\, S_k - S_{k-1} \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big] \,\Big|\, S_0 \,\Big] \\ &= \mathsf{E}_{\mathrm{rn}}\Big[\, \delta_{k-1}(S_0,...,S_{k-1}) \times \Big(\mathsf{E}_{\mathrm{rn}}\Big[\, S_k \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big] \;-\; S_{k-1} \Big) \,\Big|\, S_0 \,\Big] \end{aligned} \tag{3.11}$$

And now the decisive property of the risk neutral probability comes into play, namely:

$$\mathsf{E}_{\mathrm{rn}}\Big[\, S_k \,\Big|\, \{S_j\}_{j=0}^{k-1} \Big] \;-\; S_{k-1} \;\stackrel{(3.8)}{=}\; S_{k-1} \;-\; S_{k-1} \;=\; 0 \tag{3.12}$$

Thus also the expectation (3.11) vanishes and therefore the whole sum on the right hand side of (3.10) goes away if we take an expectation with respect to the risk neutral probability. Hence we end up with the compact pricing formula

$$V_0 \;=\; \mathsf{E}_{\rm rn}\big[\, H(S_0, S_1, ..., S_N)\,\big] \tag{3.13}$$

This concept generalizes to non zero interest rates and we summarize the result in the following

Theorem 3.1: Consider a price process $S_k = S(t_k)$ given by the Binomial model (3.1). Let $r \geq 0$ be the interest rates and denote by

$$s_k \;=\; e^{-r(t_k - t_0)}\, S_k \tag{3.14}$$

the discounted price process. Then the following statements hold:

a) Define the risk neutral probability (let us assume that $\Delta t := t_{k+1} - t_k$ is constant for all t_k to make the following quantity actually independent of k)

$$p_{\rm rn} \;=\; p_{\rm risk\ neutral} \;\;:=\;\; \frac{e^{r(t_{k+1}-t_k)} - 1 - {\rm ret}_{\rm down}}{{\rm ret}_{\rm up} - {\rm ret}_{\rm down}} \tag{3.15}$$

and denote expectations with respect to this probability by $\mathsf{E}_{\rm rn}[\;\cdot\;]$. Then the discounted price process $\{s_k\}_{k=0}^N$ is a martingale with respect to the risk neutral expectation. That is, the following equation holds for all $k = 0, 1, 2, ..., N-1$:

$$\mathsf{E}_{\rm rn}\big[\, s_{k+1} \,|\, \{s_j\}_{j=0}^k \,\big] \;=\; s_k \tag{3.16}$$

b) Let $H = H(S_0, S_1, ..., S_N)$ be the payoff of some option. Then the theoretical fair value of this option can be obtained from the following risk neutral expectation:

$$V_0 \;=\; e^{-r(t_N - t_0)}\, \mathsf{E}_{\rm rn}\big[\, H(S_0, S_1, ..., S_N)\,\big] \tag{3.17}$$

Proof: In the presence of interest rates (3.2) is substituted by

$$h(S_0, S_1, ..., S_N) \;=\; v_0 + \sum_{k=1}^{N} \delta_{k-1}(S_0, ..., S_{k-1}) \times (s_k - s_{k-1}) \tag{3.18}$$

with $h = e^{-r(t_N - t_0)} H$ being the discounted payoff function. Thus, if we want to eliminate the sum on the right hand side of (3.18) by taking an expectation value, we need to have the following property:

$$\mathsf{E}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^k \,\big] \;\stackrel{!}{=}\; s_k \tag{3.19}$$

or

$$e^{-r(t_{k+1}-t_0)}\,\mathsf{E}\big[\,S_{k+1}\,|\,\{S_j\}_{j=0}^k\,\big] \;\stackrel{!}{=}\; e^{-r(t_k-t_0)}\,S_k \tag{3.20}$$

which, using (3.7), is equivalent to

$$\begin{aligned} S_k \times \Big(1 \,+\, \mathrm{ret_{up}}\cdot p \,+\, \mathrm{ret_{down}}\cdot(1-p)\Big) \;&\stackrel{!}{=}\; e^{r(t_{k+1}-t_k)}\,S_k \\ \Leftrightarrow \quad \mathrm{ret_{up}}\cdot p \,+\, \mathrm{ret_{down}}\cdot(1-p) \;&=\; e^{r(t_{k+1}-t_k)} - 1 \\ \Leftrightarrow \quad (\mathrm{ret_{up}} - \mathrm{ret_{down}})p \;&=\; e^{r(t_{k+1}-t_k)} - 1 - \mathrm{ret_{down}} \end{aligned}$$

which gives

$$p \;=\; \frac{e^{r(t_{k+1}-t_k)} - 1 - \mathrm{ret_{down}}}{\mathrm{ret_{up}} - \mathrm{ret_{down}}} \;=:\; p_{\text{risk neutral}} \;=:\; p_{\mathrm{rn}}$$

This proves part (a). Part (b) is obtained in the same way as (3.13) is obtained through (3.10-3.12). Equation (3.12) is substituted by

$$\mathsf{E}_{\mathrm{rn}}\Big[\,s_k\,\Big|\,\{S_j\}_{j=0}^{k-1}\Big] \;-\; s_{k-1} \;\stackrel{(3.19)}{=}\; s_{k-1} \;-\; s_{k-1} \;=\; 0$$

That is, in the presence of interest rates the risk neutral probability has to be chosen such that the discounted price process $\{s_k\}_{k=0}^N$ is a martingale. Then taking expectation value of (3.18) with this risk neutral probability eliminates the sum on the right hand side of (3.18) and (3.17) follows since $v_0 = e^{-r(t_0-t_0)}V_0 = V_0$. ■

Let us also note the following

Corollary 3.1: Consider a price process $S_k = S(t_k)$ given by the Binomial model (3.1). Let $r \geq 0$ be the interest rates and denote by

$$v_k \;=\; e^{-r(t_k-t_0)}\,V_k \tag{3.21}$$

the discounted time t_k portfolio value of the replicating portfolio V_k. Then the process $\{v_k\}_{k=0}^N$ is a martingale with respect to expectations with the risk neutral probability (3.15). That is,

$$\mathsf{E}_{\mathrm{rn}}\big[\,v_{k+1}\,|\,\{S_j\}_{j=0}^k\,\big] \;=\; v_k \tag{3.22}$$

for all $k = 0, 1, 2, ..., N-1$.

Proof: According to part (b) of Theorem 1.1 we have

$$v_k \;=\; v_0 + \sum_{j=1}^k \delta_{j-1}(s_j - s_{j-1})$$

from which we get

$$v_{k+1} = v_k + \delta_k (s_{k+1} - s_k) \tag{3.23}$$

Thus, since $v_k = v_k(S_0, .., S_k)$ and $\delta_k = \delta_k(S_0, .., S_k)$ do not depend on S_{k+1}

$$\begin{aligned} \mathsf{E}_{\rm rn}\big[\, v_{k+1} \,|\, \{S_j\}_{j=0}^k \big] &= \mathsf{E}_{\rm rn}\big[\, v_k + \delta_k (s_{k+1} - s_k) \,|\, \{S_j\}_{j=0}^k \big] \\ &= v_k + \delta_k \times \mathsf{E}_{\rm rn}\big[\, s_{k+1} - s_k \,|\, \{S_j\}_{j=0}^k \big] \\ &= v_k + \delta_k \times \big(\mathsf{E}_{\rm rn}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^k \big] \,-\, s_k \big) \\ &= v_k \end{aligned}$$

where we used again the martingale property $\mathsf{E}_{\rm rn}\big[\, s_{k+1} \,|\, \{S_j\}_{j=0}^k \big] = s_k$ in the last line. ■

Remark: Equation (3.22) is actually equivalent to the recursion relation (2.14) of Theorem 2.1. To see this more explicitely, recall the definition $d_{k,k+1} = e^{-r(t_{k+1}-t_k)}$ such that (2.14) becomes

$$\begin{aligned} e^{r(t_k - t_0)} v_k \;=\; \frac{1}{\mathrm{ret_{up}} - \mathrm{ret_{down}}} \Big\{ & (1 - e^{-r(t_{k+1}-t_k)} - e^{-r(t_{k+1}-t_k)} \mathrm{ret_{down}}) e^{r(t_{k+1}-t_0)} v_{k+1}^{\rm up} \;- \\ & (1 - e^{-r(t_{k+1}-t_k)} - e^{-r(t_{k+1}-t_k)} \mathrm{ret_{up}}) e^{r(t_{k+1}-t_0)} v_{k+1}^{\rm down} \Big\} \end{aligned}$$

Multiplying the above equation with $e^{-r(t_k - t_0)} = e^{r(t_{k+1}-t_k)} e^{-r(t_{k+1}-t_0)}$ on both sides,

$$\begin{aligned} v_k &= \frac{(e^{r(t_{k+1}-t_k)} - 1 - \mathrm{ret_{down}}) v_{k+1}^{\rm up} \;-\; (e^{r(t_{k+1}-t_k)} - 1 - \mathrm{ret_{up}}) v_{k+1}^{\rm down}}{\mathrm{ret_{up}} - \mathrm{ret_{down}}} \\ &= \frac{e^{r(t_{k+1}-t_k)} - 1 - \mathrm{ret_{down}}}{\mathrm{ret_{up}} - \mathrm{ret_{down}}}\, v_{k+1}^{\rm up} \;+\; \frac{\mathrm{ret_{up}} + 1 - e^{r(t_{k+1}-t_k)}}{\mathrm{ret_{up}} - \mathrm{ret_{down}}}\, v_{k+1}^{\rm down} \\ &= p_{\rm rn} \cdot v_{k+1}^{\rm up} \;+\; (1 - p_{\rm rn}) \cdot v_{k+1}^{\rm down} \\ &= p_{\rm rn} \cdot v_{k+1}\big(S_0, ..., S_k, S_k(1 + \mathrm{ret_{up}})\big) \;+\; (1 - p_{\rm rn}) \cdot v_{k+1}\big(S_0, ..., S_k, S_k(1 + \mathrm{ret_{down}})\big) \\ &= \mathsf{E}_{\rm rn}\big[\, v_{k+1}(S_0, ..., S_{k+1}) \,|\, \{S_j\}_{j=0}^k \big] \end{aligned} \tag{3.24}$$

where the last line in (3.24) is simply the definition of the conditional expectation value $\mathsf{E}_{\rm rn}\big[\, \cdot \,|\, \{S_j\}_{j=0}^k \big]$, the variable S_{k+1} 'is integrated out' but all earlier variables $S_1, ..., S_k$ are kept fix. ■

The case of a european option $H = H(S_N)$ which depends on the underlying at maturity only is again of special interest and it allows a more explicit pricing formula:

Theorem 3.2: Consider a price process $S_k = S(t_k)$ given by the Binomial model (3.1) and let $r \geq 0$ denote the interest rates. Let p_{rn} be the risk neutral probability (3.15). Let

$$H = H(S_N) \tag{3.25}$$

be the payoff of some european option which depends on the underlying at maturity only. Then the time zero t_0 theoretical fair value V_0 of this option can be written as

$$V_0 = e^{-r(t_N - t_0)} \sum_{j=0}^{N} H\big(S_0(1+\text{ret}_{\text{up}})^j (1+\text{ret}_{\text{down}})^{N-j}\big) \times \binom{N}{j} p_{\text{rn}}^j (1-p_{\text{rn}})^{N-j} \tag{3.26}$$

More generally, the time t_k theoretical fair value V_k is given by

$$V_k = e^{-r(t_N - t_k)} \sum_{j=0}^{N-k} H\big(S_k(1+\text{ret}_{\text{up}})^j (1+\text{ret}_{\text{down}})^{N-k-j}\big) \times \binom{N-k}{j} p_{\text{rn}}^j (1-p_{\text{rn}})^{N-k-j} \tag{3.27}$$

Proof: Apparently (3.26) is the $k = 0$ version of (3.27), thus it is sufficient to prove the latter formula. We prove (3.27) by induction on k, starting with $k = N$: For $k = N$, (3.27) reduces to $V_N = H(S_N)$ which is correct. Suppose equation (3.27) is correct for $k+1$. We have to show then that it is also correct for k. From (3.24) we have the recursion

$$v_k = p_{\text{rn}} \cdot v_{k+1}\big(S_0, ..., S_k, S_k(1+\text{ret}_{\text{up}})\big) + (1-p_{\text{rn}}) \cdot v_{k+1}\big(S_0, ..., S_k, S_k(1+\text{ret}_{\text{down}})\big)$$

or

$$\begin{aligned} V_k &= p_{\text{rn}}\, e^{-r(t_{k+1}-t_k)} V_{k+1}\big(S_k(1+\text{ret}_{\text{up}})\big) + (1-p_{\text{rn}})\, e^{-r(t_{k+1}-t_k)} V_{k+1}\big(S_k(1+\text{ret}_{\text{down}})\big) \\ &\overset{\substack{\text{ind.}\\ \text{hyp.}}}{=} p_{\text{rn}}\, e^{-r(t_{k+1}-t_k)} e^{-r(t_N - t_{k+1})} \times \\ &\quad \sum_{j=0}^{N-(k+1)} H\big(S_k(1+\text{ret}_{\text{up}})(1+\text{ret}_{\text{up}})^j (1+\text{ret}_{\text{down}})^{N-(k+1)-j}\big) \times \\ &\quad \binom{N-(k+1)}{j} p_{\text{rn}}^j (1-p_{\text{rn}})^{N-(k+1)-j} \\ &\quad + (1-p_{\text{rn}})\, e^{-r(t_{k+1}-t_k)} e^{-r(t_N - t_{k+1})} \times \\ &\quad \sum_{j=0}^{N-(k+1)} H\big(S_k(1+\text{ret}_{\text{down}})(1+\text{ret}_{\text{up}})^j (1+\text{ret}_{\text{down}})^{N-(k+1)-j}\big) \times \\ &\quad \binom{N-(k+1)}{j} p_{\text{rn}}^j (1-p_{\text{rn}})^{N-(k+1)-j} \end{aligned}$$

This is equivalent to

$$
\begin{aligned}
e^{r(t_N-t_k)}V_k &= \sum_{j=0}^{N-k-1} H\big(S_k(1+\text{ret}_{\text{up}})^{j+1}(1+\text{ret}_{\text{down}})^{N-k-(j+1)}\big) \times \\
&\quad \binom{N-k-1}{j} p_{\text{rn}}^{j+1}(1-p_{\text{rn}})^{N-k-(j+1)} \\
&\quad + \sum_{j=0}^{N-k-1} H\big(S_k(1+\text{ret}_{\text{up}})^{j}(1+\text{ret}_{\text{down}})^{N-k-j}\big) \times \\
&\quad \binom{N-k-1}{j} p_{\text{rn}}^{j}(1-p_{\text{rn}})^{N-k-j}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{N-k} H\big(S_k(1+\text{ret}_{\text{up}})^{j}(1+\text{ret}_{\text{down}})^{N-k-j}\big) \times \\
&\quad \binom{N-k-1}{j-1} p_{\text{rn}}^{j}(1-p_{\text{rn}})^{N-k-j} \\
&\quad + \sum_{j=0}^{N-k-1} H\big(S_k(1+\text{ret}_{\text{up}})^{j}(1+\text{ret}_{\text{down}})^{N-k-j}\big) \times \\
&\quad \binom{N-k-1}{j} p_{\text{rn}}^{j}(1-p_{\text{rn}})^{N-k-j}
\end{aligned}
$$

$$
\begin{aligned}
&= H\big(S_k(1+\text{ret}_{\text{up}})^{N-k}\big) \times \tbinom{N-k-1}{N-k-1} p_{\text{rn}}^{N-k} \\
&\quad + \sum_{j=1}^{N-k-1} H\big(S_k(1+\text{ret}_{\text{up}})^{j}(1+\text{ret}_{\text{down}})^{N-k-j}\big) \times \\
&\quad \left[\binom{N-k-1}{j-1} + \binom{N-k-1}{j}\right] p_{\text{rn}}^{j}(1-p_{\text{rn}})^{N-k-j} \\
&\quad + H\big(S_k(1+\text{ret}_{\text{down}})^{N-k}\big) \tbinom{N-k-1}{0} (1-p_{\text{rn}})^{N-k}
\end{aligned}
$$

$$
\begin{aligned}
&= H\big(S_k(1+\text{ret}_{\text{up}})^{N-k}\big)\, p_{\text{rn}}^{N-k} \;+\; H\big(S_k(1+\text{ret}_{\text{down}})^{N-k}\big)\,(1-p_{\text{rn}})^{N-k} \\
&\quad + \sum_{j=1}^{N-k-1} H\big(S_k(1+\text{ret}_{\text{up}})^{j}(1+\text{ret}_{\text{down}})^{N-k-j}\big) \times \binom{N-k}{j} p_{\text{rn}}^{j}(1-p_{\text{rn}})^{N-k-j}
\end{aligned}
$$

$$
= \sum_{j=0}^{N-k} H\big(S_k(1+\text{ret}_{\text{up}})^{j}(1+\text{ret}_{\text{down}})^{N-k-j}\big) \times \binom{N-k}{j} p_{\text{rn}}^{j}(1-p_{\text{rn}})^{N-k-j}
$$

and this proves the theorem. ■

Finally, since the title of this chapter is "Real World and Risk Neutral Probabilities", let us actually say what the phrase 'real world' usually refers to: A real world probability for the returns $\mathrm{ret}_k \in \{\mathrm{ret}_{\mathrm{up}}, \mathrm{ret}_{\mathrm{down}}\}$ would be any probability which is obtained by some kind of estimation procedure applied to the actual realized historic prices (the real world realization of the price process). Thus, opposite to the notion of risk neutral probability, the phrase real world probability does not refer to a uniquely defined formula. As we have pointed out already several times, option prices are actually completely independent of these real world probabilities.

So far, our considerations were mainly driven by the question: How can we replicate option payoffs by a suitable trading strategy in the underlying? This led to the Binomial model: if we only allow 2 outcomes for a stock return by going from one time step to the next, exact replication is always possible. In the next section, we take a somewhat different point of view and simply ask the question: by looking at actual stock (or commodity or FX) return data, what is actually a realistic model for the returns? Apparently, we cannot just allow 2 possible outcomes by going from one day to the next, but a whole continuum of prices is possible. This will lead us to the Black-Scholes model and the basic concept of a Brownian motion. Then, following that section, we will actually see that the Black-Scholes model can be written as a continuous time limit of a suitable Binomial model and, therefore, exact replication of option payoffs in the Black-Scholes model is still possible, the Black-Scholes model is a so called complete model.