Chapter 6

Price and Greeks of Standard Options

In the last section we have seen that in the Black-Scholes model the theoretical fair value of some european option with payoff $H = H(S_T)$ is given by the one dimensional integral

$$FV_{\rm BS} \;=\; e^{-rT}\int_{\mathbb{R}} H\big(S_0 e^{\sigma\sqrt{T}x+(r-\frac{\sigma^2}{2})T}\big)e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} \tag{6.1}$$

The case of a call and a put option is of particular interest and because of their importance, we write down the corresponding formulae into this separate chapter. Before we do that, let us introduce the option sensitivities, the "greeks":

Definition 6.1: Let $H = H(\{S_t\}_{0\leq t\leq T})$ be the payoff of some (possibly path dependent) option and let

$$FV_t \;=\; FV(S_t,\sigma,r,t) \tag{6.2}$$

denote the time t theoretical fair value of H (evaluated with respect to some asset price model, not necessarily the Black-Scholes model). Here S_t is the asset price at time t, σ denotes the volatility and r denotes the interest rates. Then the following abbreviations has been become standard in the derivatives community (listed with decreasing importance):

$$\delta \;:=\; \tfrac{\partial FV_t}{\partial S_t} \qquad \text{(delta)} \tag{6.3}$$
$$vega \;:=\; \tfrac{\partial FV_t}{\partial \sigma} \qquad \text{(vega)} \tag{6.4}$$
$$\rho \;:=\; \tfrac{\partial FV_t}{\partial r} \qquad \text{(rho)} \tag{6.5}$$
$$\theta \;:=\; \tfrac{\partial FV_t}{\partial t} \qquad \text{(theta)} \tag{6.6}$$
$$\gamma \;:=\; \tfrac{\partial^2 FV_t}{\partial S_t^2} \qquad \text{(gamma)} \tag{6.7}$$

Now let us summarize the formulae for call and put options:

Theorem 6.1: Consider a call and a put option with strike K and maturity T,

$$H_{\text{call}} = \max\{ S_T - K \,,\, 0 \} \tag{6.8}$$

$$H_{\text{put}} = \max\{ K - S_T \,,\, 0 \} \tag{6.9}$$

and let $FV^{\text{BS}}_{\text{call},t}$ and $FV^{\text{BS}}_{\text{put},t}$ denote their time t theoretical fair values in the Black-Scholes model. Let

$$N(x) := \int_{-\infty}^{x} e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} \tag{6.10}$$

denote the cumulative normal distribution function and define the quantities

$$d_{\pm} := \frac{\log \frac{S_t}{K} + (r \pm \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{6.11}$$

such that $d_+ - d_- = \sigma\sqrt{T-t}$. Then there are the following formulae:

Fair Value:

$$FV^{\text{BS}}_{\text{call},t} = S_t N(d_+) - K e^{-r(T-t)} N(d_-) \tag{6.12}$$

$$FV^{\text{BS}}_{\text{put},t} = -S_t N(-d_+) + K e^{-r(T-t)} N(-d_-) \tag{6.13}$$

Delta:

$$\delta_{\text{call}} = N(d_+) \tag{6.14}$$

$$\delta_{\text{put}} = -N(-d_+) \tag{6.15}$$

Vega:

$$vega_{\text{call}} = S_t N'(d_+)\sqrt{T-t} \tag{6.16}$$

$$vega_{\text{put}} = vega_{\text{call}} \tag{6.17}$$

Rho:

$$\rho_{\text{call}} = K(T-t)e^{-r(T-t)} N(d_-) \tag{6.18}$$

$$\rho_{\text{put}} = -K(T-t)e^{-r(T-t)} N(-d_-) \tag{6.19}$$

Theta:

$$\theta_{\text{call}} = -S_t \frac{\sigma}{2\sqrt{T-t}} N'(d_+) - rKe^{-r(T-t)} N(d_-) \tag{6.20}$$

$$\theta_{\text{put}} = -S_t \frac{\sigma}{2\sqrt{T-t}} N'(-d_+) + rKe^{-r(T-t)} N(-d_-) \tag{6.21}$$

Gamma:

$$\gamma_{\text{call}} = \frac{N'(d_+)}{\sigma\sqrt{T-t}S_t} \tag{6.22}$$
$$\gamma_{\text{put}} = \gamma_{\text{call}} \tag{6.23}$$

Proof: In the Black-Scholes model, the time t theoretical fair value of some european option with maturity $T \geq t$ and payoff $H = H(S_T)$ is given by

$$FV_t^{\text{BS}} = e^{-r(T-t)} \int_{\mathbb{R}} H\big(S_t e^{\sigma\sqrt{T-t}\,x + (r-\frac{\sigma^2}{2})(T-t)}\big) e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \tag{6.24}$$

Let us abbreviate

$$\tau := T - t \tag{6.25}$$
$$a := \sigma\sqrt{\tau} \tag{6.26}$$
$$b := (r - \sigma^2/2)\tau \tag{6.27}$$

and define x_0 as the solution of the equation

$$S_t e^{\sigma\sqrt{T-t}x + (r-\frac{\sigma^2}{2})(T-t)} = S_t e^{ax+b} \overset{!}{=} K \tag{6.28}$$

That is,

$$x_0 = \tfrac{1}{a}\log(K/S_t) - \tfrac{b}{a} \tag{6.29}$$

Then

$$\begin{aligned}
FV_{\text{call},t}^{\text{BS}} &= e^{-r\tau} \int_{\mathbb{R}} H_{\text{call}}\big(S_t e^{a\,x+b}\big) e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau} \int_{\mathbb{R}} \max\big\{\, S_t e^{a\,x+b} - K\,,\, 0\,\big\} e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau} \int_{x_0}^{\infty} \big(\, S_t e^{a\,x+b} - K\,\big) e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau} S_t\, e^{b+\frac{a^2}{2}} \int_{x_0}^{\infty} e^{-\frac{1}{2}(x-a)^2} \tfrac{dx}{\sqrt{2\pi}} - e^{-r\tau} K \int_{x_0}^{\infty} e^{-\frac{x^2}{2}} \tfrac{dx}{\sqrt{2\pi}} \\
&= e^{-r\tau} S_t\, e^{b+\frac{a^2}{2}} N(-x_0 + a) - e^{-r\tau} K N(-x_0) \\
&= S_t\, N(d_+) - e^{-r(T-t)} K N(d_-)
\end{aligned} \tag{6.30}$$

since

$$-x_0 = \tfrac{1}{a}\log(S_t/K) + \tfrac{b}{a} = \tfrac{\log(S_t/K) + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}} = d_-$$

and $-x_0 + a = d_- + \sigma\sqrt{\tau} = d_+$. Furthermore

$$-r\tau + b + a^2/2 \;=\; -r\tau + (r - \sigma^2/2)\tau + (\sigma\sqrt{\tau})^2/2 \;=\; 0$$

which results in the last line of (6.30). The proof for the put fair value is similar. In particular, since

$$N(d) + N(-d) \;=\; 1$$

we get from (6.12,6.13)

$$FV^{\mathrm{BS}}_{\mathrm{call},t} - FV^{\mathrm{BS}}_{\mathrm{put},t} \;=\; S_t - e^{-r(T-t)}K \tag{6.31}$$

This relation is called 'call-put parity'.

Delta: To compute the delta we use the third line of (6.30) instead of the last one. We get

$$\begin{aligned}
\delta_{\mathrm{call}}(S_t,t) &= \frac{\partial}{\partial S_t}\Big\{e^{-r\tau}\int_{x_0}^{\infty}(S_t\,e^{ax+b} - K)\,e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}}\Big\} \\
&= e^{-r\tau}\int_{x_0}^{\infty} e^{ax+b}\,e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \;-\; e^{-r\tau}\underbrace{(S_t\,e^{ax_0+b} - K)}_{=0}\,\tfrac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}}\,\tfrac{\partial x_0}{\partial S_t} \\
&= N(d_+)
\end{aligned} \tag{6.32}$$

since the above integral is the same as the first integral in (6.30), up to the factor of S_t which has been differentiated away. Finally it follows from call-put parity that

$$\delta_{\mathrm{put}}(S,t) = N(d_+) - 1 = -N(-d_+)\ .$$

Vega: Again we use the third line of (6.30):

$$\begin{aligned}
vega_{\mathrm{call}}(S_t,t) &= \frac{\partial}{\partial \sigma}\Big\{e^{-r\tau}\int_{x_0}^{\infty}(S_t\,e^{ax+b} - K)\,e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}}\Big\} \\
&= e^{-r\tau}\int_{x_0}^{\infty} S_t\big(\tfrac{\partial a}{\partial\sigma}x + \tfrac{\partial b}{\partial\sigma}\big)e^{ax+b}\,e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \;-\; e^{-r\tau}\underbrace{(S_t\,e^{ax_0+b} - K)}_{=0}\,\tfrac{e^{-\frac{x_0^2}{2}}}{\sqrt{2\pi}}\,\tfrac{\partial x_0}{\partial\sigma} \\
&= e^{-r\tau}e^{b+\frac{a^2}{2}}\,S_t\int_{x_0}^{\infty}(\sqrt{\tau}x - \sigma\tau)\,e^{-\frac{(x-a)^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \\
&= S_t\sqrt{\tau}\int_{x_0}^{\infty}(x-a)\,e^{-\frac{(x-a)^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \\
&= S_t\sqrt{\tau}\Big[-\frac{e^{-\frac{(x-a)^2}{2}}}{\sqrt{2\pi}}\Big]_{x_0}^{\infty} \;=\; S_t\sqrt{\tau}\,\frac{e^{-\frac{(x_0-a)^2}{2}}}{\sqrt{2\pi}} \\
&= S_t\sqrt{\tau}\,\frac{e^{-\frac{(-d_+)^2}{2}}}{\sqrt{2\pi}} \;=\; S_t\sqrt{\tau}\,\frac{e^{-\frac{d_+^2}{2}}}{\sqrt{2\pi}} \;=\; S_t\sqrt{\tau}\,N'(d_+)
\end{aligned} \tag{6.33}$$

The vega for the put follows again from call-put parity. The remaining formulae may be looked up, for example, in the book on Quantitative Finance, Vol.1, by Paul Wilmott. ■