Chapter 4

The Standard Asset Price Model

Consider some discrete times t_k in the intervall $[0, T]$,

$$t_k = k\frac{T}{N} = k\Delta t\,, \quad k = 0, 1, ..., N \tag{4.1}$$

where

$$N = N_T = \frac{T}{\Delta t} \in \mathbb{N} \tag{4.2}$$

Let $S_{t_k} = S_{k\Delta t}$ be the price of some stock at time t_k and denote the returns by going from one time step to the next by

$$\mathrm{ret}_{t_k} \;=\; \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \tag{4.3}$$

One may think of Δt being one day and S_{t_k} being the closing prices at each day although later we will consider the limit $\Delta t \to 0$. It is an empirical fact that the daily returns of many assets are bell shaped, like a Gaussian distribution. Thus, as a first approximation, one may write down the following stochastic model ($k = 0, 1, ..., N_T - 1$)

$$\mathrm{ret}_{t_k} \;=\; \text{mean} \;+\; \text{standard deviation} \times \phi_k \tag{4.4}$$

where the ϕ_k are identically independent normally distributed random variables with mean zero and variance one,

$$\phi_k \in \mathcal{N}(0, 1) \quad \text{i.i.d.} \tag{4.5}$$

This is only a first approximation. There are deviations from a Gaussian distribution. Most financial data have more heavy tails than a normal distribution and a higher peak at the mean value. Furthermore, the returns in (4.4) are not completely independent. Many financial data show a positive correlation of the absolute values of the returns, of $|\mathrm{ret}_{t_k}|$ and $|\mathrm{ret}_{t_{k+m}}|$. In the book of Shiryaev [17] one can find a detailed discussion of the statistical analysis of financial data (in Chapter 4) as well as an overview of the proposed stochastic models to fit these data.

We now analyze how the mean and the standard deviation in (4.4) have to scale with Δt in order to get a reasonable model in the time continuous case $\Delta t \to 0$. To this end we write

$$\mathrm{ret}_{t_k} \;=\; \mu\,\Delta t^{\alpha} + \sigma\,\Delta t^{\beta}\,\phi_k \tag{4.6}$$

such that

$$S_{t_k} \;=\; S_{t_{k-1}}(1 + \mu\,\Delta t^{\alpha} + \sigma\,\Delta t^{\beta}\phi_k)$$

or, with $t = N_t \times \Delta t$, $N_t = t/\Delta t$,

$$S_t = S_0 \prod_{k=1}^{N_t} \left(1 + \mu\,\Delta t^{\alpha} + \sigma\,\Delta t^{\beta}\phi_k\right) \tag{4.7}$$

Suppose for the moment the model is deterministic, $\sigma = 0$. Then, using the first order Taylor expansion $\log(1+x) = x + O(x^2)$ in the third line,

$$\begin{aligned} S_t &= S_0\,(1+\mu\,\Delta t^{\alpha})^{N_t} \\ &= S_0\,e^{N_t \log(1+\mu\,\Delta t^{\alpha})} \\ &= S_0\,e^{N_t(\mu\Delta t^{\alpha} + O(\Delta t^{2\alpha}))} \\ &= S_0\,e^{\mu t \Delta t^{\alpha-1} + O(\Delta t^{2\alpha-1})} \end{aligned} \tag{4.8}$$

which gives $\alpha = 1$ and exponential growth (or decrease) in the time continuous case, $S_t = S_0\,e^{\mu t}$ which is simply the solution of $dS/S = \mu dt$. Now consider the stochastic part in (4.6). For simplicity, we put $\mu = 0$. Then, now using the second order Taylor expansion $\log(1+x) = x - x^2/2 + O(x^3)$ in the third line,

$$\begin{aligned} S_t &= S_0 \prod_{k=1}^{N_t} \left(1 + \sigma\,\Delta t^{\beta}\phi_k\right) \\ &= S_0\,e^{\sum_{k=1}^{N_t} \log(1+\sigma\,\Delta t^{\beta}\phi_k)} \\ &= S_0\,e^{\sum_{k=1}^{N_t}\left(\sigma\,\Delta t^{\beta}\phi_k - \frac{1}{2}\sigma^2\Delta t^{2\beta}\phi_k^2 + O(\Delta t^{3\beta})\right)} \\ &= S_0\,e^{\sigma\,\Delta t^{\beta}\sum_{k=1}^{N_t}\phi_k - \frac{\sigma^2}{2}\Delta t^{2\beta}\sum_{k=1}^{N_t}\phi_k^2 + O(N_t\Delta t^{3\beta} = \Delta t^{3\beta-1})} \end{aligned} \tag{4.9}$$

We now consider for what values of β the expectation

$$\mathsf{E}\left[f\left(\Delta t^{\beta}\textstyle\sum_{k=1}^{N_t}\phi_k\right)\right] \;=\; \displaystyle\int_{\mathbb{R}^{N_t}} f\left(\Delta t^{\beta}\textstyle\sum_{k=1}^{N_t}\phi_k\right) \displaystyle\prod_{k=1}^{N_t} \frac{1}{\sqrt{2\pi}}\,e^{-\frac{\phi_k^2}{2}}\,d\phi_k \tag{4.10}$$

has a nontrivial limit. Here f is some function. We make a substitution of variables $(\phi_k)_{1\le k\le N_t} \to (x_k)_{1\le k\le N_t}$ defined as follows:

$$\begin{array}{rcl} x_1 &=& \sqrt{\Delta t}\,\phi_1 \\ x_2 &=& \sqrt{\Delta t}\,(\phi_1+\phi_2) \\ x_3 &=& \sqrt{\Delta t}\,(\phi_1+\phi_2+\phi_3) \\ \vdots && \\ x_{N_t} &=& \sqrt{\Delta t}\,(\phi_1+\phi_2+\ldots+\phi_{N_t}) \end{array} \quad\Leftrightarrow\quad \begin{array}{rcl} \phi_1 &=& x_1/\sqrt{\Delta t} \\ \phi_2 &=& (x_2-x_1)/\sqrt{\Delta t} \\ \phi_3 &=& (x_3-x_2)/\sqrt{\Delta t} \\ \vdots && \\ \phi_{N_t} &=& (x_{N_t}-x_{N_t-1})/\sqrt{\Delta t} \end{array} \tag{4.11}$$

and instead of labelling the x with $k \in \{1, 2, ..., N_t\}$, we label them with $k\Delta t$ which has the meaning of time. In particular, $N_t \Delta t = t$. So, rename $x_k \to x_{k\Delta t}$. The Jacobian of the transformation (4.11) is $\det \frac{\partial \phi}{\partial x} = 1/\sqrt{\Delta t}^{N_t}$. The expectation (4.10) becomes

$$\mathsf{E}\left[f\left(\Delta t^{\beta} \sum_{k=1}^{N_t} \phi_k\right)\right] = \int_{\mathbb{R}^{N_t}} f\left(\Delta t^{\beta - \frac{1}{2}} x_t\right) \prod_{k=1}^{N_t} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \tag{4.12}$$

where we introduced the kernel

$$p_\tau(x, y) := \frac{1}{\sqrt{2\pi\tau}}\, e^{-\frac{(x-y)^2}{2\tau}} \tag{4.13}$$

and used the definition

$$x_0 := 0 \tag{4.14}$$

The kernel (4.13) has the following basic property:

Lemma 4.1: *Let $p_t(x, y)$ be given by (4.13). Then*

$$\int_{\mathbb{R}} p_s(x, y) p_t(y, z)\, dy = p_{s+t}(x, z) \tag{4.15}$$

Proof: We have

$$\begin{aligned}
p_s(x, y) p_t(y, z) &= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}}\, e^{-(\frac{1}{2s} + \frac{1}{2t}) y^2 + (\frac{x}{s} + \frac{z}{t}) y} \\
&= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}}\, e^{-\frac{s+t}{2st}(y^2 - 2\frac{xt+zs}{s+t} y)} \\
&= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}}\, e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} e^{\frac{s+t}{2st}(\frac{xt+zs}{s+t})^2} \\
&= \frac{e^{-\frac{x^2}{2s} - \frac{z^2}{2t}}}{2\pi\sqrt{st}}\, e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} e^{\frac{(xt+zs)^2}{2st(s+t)}} \\
&= \frac{1}{2\pi\sqrt{st}}\, e^{-x^2(\frac{1}{2s} - \frac{t}{2s(s+t)}) - z^2(\frac{1}{2t} - \frac{s}{2t(s+t)}) + \frac{xz}{s+t}}\; e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}}\, e^{-\frac{x^2}{2(s+t)} - \frac{z^2}{2(s+t)} + \frac{xz}{s+t}}\; e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} \\
&= \frac{1}{2\pi\sqrt{st}}\, e^{-\frac{(x-z)^2}{2(s+t)}}\; e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2}
\end{aligned} \tag{4.16}$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}} p_s(x, y) p_t(y, z)\, dy &= \frac{1}{2\pi\sqrt{st}}\, e^{-\frac{(x-z)^2}{2(s+t)}} \int_{\mathbb{R}} e^{-\frac{s+t}{2st}(y - \frac{xt+zs}{s+t})^2} dy \\
&= \frac{1}{2\pi\sqrt{st}}\, e^{-\frac{(x-z)^2}{2(s+t)}} \int_{\mathbb{R}} e^{-\frac{s+t}{2st} v^2} dv \\
&= \frac{1}{2\pi\sqrt{st}}\, e^{-\frac{(x-z)^2}{2(s+t)}} \sqrt{2\pi} \sqrt{\frac{st}{s+t}} \\
&= p_{s+t}(x, z)
\end{aligned}$$

which proves the lemma. ■

Using this lemma, (4.12) simplifies to

$$\begin{aligned}\mathsf{E}\Big[f\Big(\Delta t^{\beta}\textstyle\sum_{k=1}^{N_t}\phi_k\Big)\Big] &= \int_{\mathbb{R}} f\big(\Delta t^{\beta-\frac{1}{2}}\, x_t\big) p_{N_t\Delta t}(x_0, x_t)\, dx_t \\ &= \int_{\mathbb{R}} f\big(\Delta t^{\beta-\frac{1}{2}}\, x_t\big)\, \tfrac{1}{\sqrt{2\pi t}}\, e^{-\frac{x_t^2}{2t}}\, dx_t \qquad (4.17)\end{aligned}$$

since $x_0 = 0$. Hence, a nontrivial limit is obtained only for $\beta = \frac{1}{2}$.

Definition 4.1: *Let $N_T = T/\Delta t$. The measure*

$$dW(\{x_t\}_{0<t\le T}) := \lim_{\Delta t\to 0} \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \qquad (4.18)$$

is called the Wiener measure and the family of random variables $\{x_t\}_{0<t\le T}$ is called a Brownian motion. In terms of i.i.d. random variables $\phi_k \in \mathcal{N}(0,1)$,

$$x_t = \lim_{\Delta t\to 0} \sqrt{\Delta t} \sum_{k=1}^{t/\Delta t} \phi_k \qquad (4.19)$$

Integrals with respect to the Wiener measure are computed according to the following

Theorem 4.1: *Let $F : \mathbb{R}^m \to \mathbb{R}$ be some function and let $0 =: t_0 < t_1 < \cdots < t_m \le T$. Then*

$$\int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\le T}) = \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell} \qquad (4.20)$$

Proof: Because of (4.15) only the $x_{t_1}, ..., x_{t_m}$ integration variables survive:

$$\begin{aligned}\int F(x_{t_1}, ..., x_{t_m})\, dW(\{x_t\}_{0<t\le T}) &= \lim_{\Delta t\to 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, ..., x_{t_m}) \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \\ &= \lim_{\Delta t\to 0} \int_{\mathbb{R}^{N_T}} F(x_{t_1}, ..., x_{t_m}) \prod_{k=1}^{N_{t_1}} p_{\Delta t}(...)\, dx_{k\Delta t} \prod_{k=N_{t_1}+1}^{N_{t_2}} p_{\Delta t}(...)\, dx_{k\Delta t} \times \cdots \\ &\qquad \cdots \times \prod_{k=N_{t_{m-1}}+1}^{N_{t_m}} p_{\Delta t}(...)\, dx_{k\Delta t} \\ &\overset{(4.15)}{=} \lim_{\Delta t\to 0} \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m})\, p_{t_1}(x_0, x_{t_1})\, dx_{t_1}\; p_{t_2-t_1}(x_{t_1}, x_{t_2})\, dx_{t_2} \;\times \cdots \\ &\qquad \cdots \times\; p_{t_m - t_{m-1}}(x_{t_{m-1}}, x_{t_m})\, dx_{t_m} \\ &= \int_{\mathbb{R}^m} F(x_{t_1}, ..., x_{t_m}) \prod_{\ell=1}^{m} p_{t_\ell - t_{\ell-1}}(x_{t_{\ell-1}}, x_{t_\ell})\, dx_{t_\ell}\end{aligned}$$

which coincides with (4.20). ■

Now we return to (4.9) and put $\beta = \frac{1}{2}$. Then

$$S_t \;=\; S_0\, e^{\sigma\sqrt{\Delta t}\sum_{k=1}^{N_t}\phi_k - \frac{\sigma^2}{2}\Delta t\sum_{k=1}^{N_t}\phi_k^2 + O(\sqrt{\Delta t})} \tag{4.21}$$

The first term in the exponent converges to a Brownian motion $x_t = \lim_{\Delta t\to 0}\sqrt{\Delta t}\,\sum_{k=1}^{N_t}\phi_k$ and the last term vanishes, but what about the second term? To this end we consider the expectation

$$\begin{aligned} \mathsf{E}\Big[f\Big(\Delta t\textstyle\sum_{k=1}^{N_t}\phi_k^2\Big)\Big] &= \int_{\mathbb{R}^{N_t}} f\Big(\Delta t\textstyle\sum_{k=1}^{N_t}\phi_k^2\Big)\prod_{k=1}^{N_t}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{\phi_k^2}{2}}\,d\phi_k \\ &= \frac{\int_{\mathbb{R}^{N_t}} f\Big(\Delta t\sum_{k=1}^{N_t}\phi_k^2\Big)e^{-\sum_{k=1}^{N_t}\frac{\phi_k^2}{2}}\,d\phi_k}{\int_{\mathbb{R}^{N_t}} e^{-\sum_{k=1}^{N_t}\frac{\phi_k^2}{2}}\,d\phi_k} \end{aligned} \tag{4.22}$$

Introducing the radial coordinate

$$\rho^2 = \textstyle\sum_{k=1}^{N_t}\phi_k^2 \tag{4.23}$$

we get

$$\begin{aligned} \mathsf{E}\Big[f\Big(\Delta t\textstyle\sum_{k=1}^{N_t}\phi_k^2\Big)\Big] &= \frac{\int_{\mathbb{R}} f(\Delta t\,\rho^2)e^{-\frac{\rho^2}{2}}\,\rho^{N_t-1}d\rho}{\int_{\mathbb{R}} e^{-\frac{\rho^2}{2}}\,\rho^{N_t-1}d\rho} \\ &\overset{v=\Delta t\rho^2}{=} \frac{\int_{\mathbb{R}} f(v)e^{-\frac{v}{2\Delta t}}\,v^{\frac{N_t}{2}-1}dv}{\int_{\mathbb{R}} e^{-\frac{v}{2\Delta t}}\,v^{\frac{N_t}{2}-1}dv} \\ &= \frac{\int_{\mathbb{R}} f(v)e^{-\frac{1}{2\Delta t}(v-t\log v+2\Delta t\log v)}\,dv}{\int_{\mathbb{R}} e^{-\frac{1}{2\Delta t}(v-t\log v+2\Delta t\log v)}\,dv} \\ &\overset{\Delta t\to 0}{\longrightarrow} f(v_{\min}) = f(t) \end{aligned} \tag{4.24}$$

where $v_{\min} = t$ is the global minimum of the exponent $v - t\log v$. In particular, the probability that $\Delta t\sum_{k=1}^{N_t}\phi_k^2$ deviates from t by at least $\varepsilon > 0$ is given by

$$\begin{aligned} \mathsf{P}\Big(\big|\Delta t\textstyle\sum_{k=1}^{N_t}\phi_k^2 - t\big| \geq \varepsilon\Big) &= \mathsf{E}\Big[\chi\big(\big|\Delta t\textstyle\sum_{k=1}^{N_t}\phi_k^2 - t\big| \geq \varepsilon\big)\Big] \\ &= \frac{\int_{\mathbb{R}} \chi\big(|v-t| \geq \varepsilon\big)\,e^{-\frac{1}{2\Delta t}(v-t\log v+2\Delta t\log v)}\,dv}{\int_{\mathbb{R}} e^{-\frac{1}{2\Delta t}(v-t\log v+2\Delta t\log v)}\,dv} \\ &\overset{\Delta t\to 0}{\longrightarrow} 0 \end{aligned} \tag{4.25}$$

as $\Delta t \to 0$. Thus we conclude[1] that the quantity

$$\lim_{\Delta t \to 0} \Delta t \sum_{k=1}^{t/\Delta t} \phi_k^2 \;=\; t \quad \text{with probability 1} \tag{4.26}$$

becomes actually deterministic in the limit $\Delta t \to 0$. In terms of the x-variables, the Brownian motions, this reads, since $\phi_k = (x_{k\Delta t} - x_{(k-1)\Delta t})/\sqrt{\Delta t}$,

$$\lim_{\Delta t \to 0} \sum_{k=1}^{t/\Delta t} (x_{k\Delta t} - x_{(k-1)\Delta t})^2 \;=\; t \quad \text{with probability 1}\,. \tag{4.27}$$

We summarize our results: The statistics of financial data suggests, as a first approximation, the stochastic model (4.4). A meaningful continuous time model is only obtained if the exponents in (4.6) are choosen to be $\alpha = 1$ and $\beta = \frac{1}{2}$ which results in

$$\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \;=\; \frac{\Delta S_{t_k}}{S_{t_{k-1}}} \;=\; \mu\,\Delta t + \sigma\,\sqrt{\Delta t}\,\phi_k \tag{4.28}$$

In view of (4.11), in particular, the right hand side thereof, and recalling the relabelling $x_k \to x_{k\Delta t}$, we may write this as

$$\frac{\Delta S_{t_k}}{S_{t_{k-1}}} \;=\; \mu\,\Delta t + \sigma\,(x_{t_k} - x_{t_{k-1}}) = \mu\,\Delta t + \sigma\,\Delta x_{t_k} \tag{4.29}$$

or, in the continuous time limit $\Delta t \to 0$,

$$\frac{dS}{S} \;=\; \mu\,dt + \sigma\,dx_t \tag{4.30}$$

where $\{x_t\}_{0<t\le T}$ is a Brownian motion. The asset price model (4.30) is called the **Black-Scholes model**. And we saw, that the discrete time solution (4.21) of (4.29) (for $\mu \neq 0$ there is an additional $\mu N_t \Delta t$ in the exponent, as in (4.8)) converges to

$$S_t = S_0\, e^{\mu t + \sigma x_t - \frac{\sigma^2}{2} t} \tag{4.31}$$

The solution (4.31) is usually refered to as a 'geometric Brownian motion'. In a following section we will rederive (4.31) as an application of the Ito-Lemma.

[1](4.25) is refered to as convergence in probability whereas (4.26) is refered to as almost sure convergence, both statements hold.