Chapter 5

The Black-Scholes Model as Continuous Time Limit of the Binomial Model

The Black-Scholes model is given by the price dynamics

$$\tfrac{dS}{S} \;=\; \mu\, dt + \sigma\, dx_t \tag{5.1}$$

with a constant drift $\mu \in \mathbb{R}$ (for example, $\mu = 5\%$), a constant volatility $\sigma > 0$ (for example, $\sigma = 20\%$) and x_t being a Brownian motion. In discrete time, this reads

$$\begin{aligned} \tfrac{\Delta S_{t_k}}{S_{t_{k-1}}} \;=\; \tfrac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}} \;&=\; \mu\, \Delta t + \sigma \Delta x_{t_k} \\ &=\; \mu\, \Delta t + \sigma (x_{t_k} - x_{t_{k-1}}) \\ &=\; \mu\, \Delta t + \sigma \sqrt{\Delta t}\, \phi_k \end{aligned} \tag{5.2}$$

with independent Gaussian $\mathcal{N}(0,1)$ random variables ϕ_k which have mean 0 and variance 1. An equivalent formulation is

$$S_{t_k} \;=\; S_{t_{k-1}} \big(1 + \mu\, \Delta t + \sigma \sqrt{\Delta t}\, \phi_k\big) \tag{5.3}$$

Recall that the dynamics of the Binomial model is given by

$$S_{t_k} \;=\; S_{t_{k-1}} \times \begin{cases} (1 + \mathrm{ret_{up}}) & \text{with some probability } \; p \\ (1 + \mathrm{ret_{down}}) & \text{with probability } \; 1 - p \end{cases} \tag{5.4}$$

Thus, in order to approximate the Black-Scholes model with a Binomial model, the following choice looks quite natural:

$$S_{t_k} \;=\; S_{t_{k-1}} \big(1 + \mu\, \Delta t + \sigma \sqrt{\Delta t}\, \varepsilon_k\big) \tag{5.5}$$

where the $\varepsilon_k \in \{+1, -1\}$ are independent random variables with (real world) probabilities

$$\mathsf{P}(\varepsilon_k = +1) \;=\; \mathsf{P}(\varepsilon_k = -1) \;=\; \frac{1}{2} \tag{5.6}$$

such that, as for the ϕ_k, we have $\mathsf{E}[\varepsilon_k] = 0$ and $\mathsf{V}[\varepsilon_k] = 1$. Observe that in this context we are not considering any options or replicating strategies. Thus, interest rates do not enter the stage here and the notion of risk neutral probabilities is not relevant at this place. Expectation values, which one could call real world expectation values in this context, with respect to the Binomial model (5.5,5.6) are given by ($t = N\Delta t$, $N = t/\Delta t$)

$$\mathsf{E}^{\mathrm{Bin}}\big[\, f(S_t)\,\big] \;=\; \sum_{k=0}^{N} f\big(S_0(1+\mu\Delta t+\sigma\sqrt{\Delta t})^k(1+\mu\Delta t-\sigma\sqrt{\Delta t})^{N-k}\big) \times \tfrac{1}{2^N}\tbinom{N}{k} \tag{5.7}$$

whereas expectation values with respect to the Black-Scholes model (5.1) are given by

$$\begin{aligned} \mathsf{E}^{\mathrm{BS}}\big[\, f(S_t)\,\big] &= \int f\big(S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma x_t}\big)\, dW(\{x_t\}_{0<t\le T}) \\ &= \int_{\mathbb{R}} f\big(S_0 e^{(\mu-\frac{\sigma^2}{2})t+\sigma x_t}\big)\, \tfrac{1}{\sqrt{2\pi t}} e^{-\frac{x_t^2}{2t}}\, dx_t \end{aligned} \tag{5.8}$$

There is the following

Theorem 5.1: In the continuous time limit $\Delta t \to 0$, the Binomial model (5.5,5.6) converges to the Black-Scholes model (5.1), we have for any square integrable function f

$$\lim_{\Delta t\to 0} \mathsf{E}^{\mathrm{Bin}}\big[\, f(S_t)\,\big] \;=\; \mathsf{E}^{\mathrm{BS}}\big[\, f(S_t)\,\big] \tag{5.9}$$

For the proof, we need the following fact:

Lemma 5.1: Let t be a fixed time, $t = N\Delta t$, $N = t/\Delta t$ and let f be some square integrable function. Then:

$$\lim_{\substack{\Delta t\to 0 \\ t \text{ fixed}}} \sum_{k=0}^{N} f\big[\sqrt{\Delta t}(2k-N)\big]\, \tfrac{1}{2^N}\tbinom{N}{k} \;=\; \int_{\mathbb{R}} f(x)\, \tfrac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}\, dx \tag{5.10}$$

Proof: Using the Fourier transform

$$f(x) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ipx}\, \hat{f}(p)\, dp \tag{5.11}$$

we obtain

$$\sum_{k=0}^{N} f\big[\sqrt{\Delta t}(2k-N)\big]\, \tfrac{1}{2^N}\tbinom{N}{k} \;=\; \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sum_{k=0}^{N} \tfrac{1}{2^N}\tbinom{N}{k} e^{ip\sqrt{\Delta t}\,(2k-N)}\, \hat{f}(p)\, dp \tag{5.12}$$

The sum in (5.12) becomes

$$\begin{aligned}
\sum_{k=0}^{N} \tfrac{1}{2^N} \binom{N}{k} e^{ip\sqrt{\Delta t}\,(2k-N)} &= e^{-ip\sqrt{\Delta t}\,N}\,\tfrac{1}{2^N}\sum_{k=0}^{N}\binom{N}{k}\; e^{2ip\sqrt{\Delta t}\,k} \\
&= e^{-ip\sqrt{\Delta t}\,N}\,\tfrac{1}{2^N}\left(1+e^{2ip\sqrt{\Delta t}}\right)^N \\
&= \tfrac{1}{2^N}\left(e^{-ip\sqrt{\Delta t}}+e^{ip\sqrt{\Delta t}}\right)^N \\
&= \left\{\cos(p\sqrt{\Delta t})\right\}^{\frac{t}{\Delta t}} \\
&= \left\{1-\tfrac{p^2\Delta t}{2}+O\big((\Delta t)^2\big)\right\}^{\frac{t}{\Delta t}} \\
&\overset{\Delta t\to 0}{\to} e^{-\frac{p^2}{2}t}
\end{aligned} \tag{5.13}$$

Thus, for $\Delta t \to 0$ (5.12) becomes

$$\begin{aligned}
\tfrac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-\frac{p^2}{2}t}\,\hat f(p)\,dp &= \tfrac{1}{\sqrt{2\pi}}\sqrt{\tfrac{1}{t}}\int_{\mathbb{R}} e^{-\frac{x^2}{2t}}\,f(x)\,dx \\
&= \tfrac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} e^{-\frac{x^2}{2t}}\,f(x)\,dx
\end{aligned} \tag{5.14}$$

where we used the fact that

$$\left(e^{-\alpha\frac{x^2}{2}}\right)^{\wedge}(p) = \tfrac{1}{\sqrt{\alpha}}\,e^{-\frac{1}{\alpha}\frac{p^2}{2}} \tag{5.15}$$

and that the Fourier transform is unitary, that is, $\int f\,g = \int \hat f\,\hat g$. ■

Proof of Theorem 5.1: We have $\log(1+x) = x - x^2/2 + O(x^3)$ and ignore terms $(\Delta t)^{\frac{3}{2}}$ in the calculation below or higher powers, since for those terms, after multiplication with $N = t/\Delta t$, at least a factor $\sqrt{\Delta t}$ survives which goes to zero in the continuous time limit $\Delta t \to 0$:

$$(1+\mu\Delta t+\sigma\sqrt{\Delta t})^k(1+\mu\Delta t-\sigma\sqrt{\Delta t})^{N-k} \tag{5.16}$$

$$\begin{aligned}
&= e^{k\log(1+\mu\Delta t+\sigma\sqrt{\Delta t})+(N-k)\log(1+\mu\Delta t-\sigma\sqrt{\Delta t})} \\
&= e^{k(\mu\Delta t+\sigma\sqrt{\Delta t}-\frac{\sigma^2}{2}\Delta t)+(N-k)(\mu\Delta t-\sigma\sqrt{\Delta t}-\frac{\sigma^2}{2}\Delta t)+O(\sqrt{\Delta t})} \\
&= e^{\sigma\sqrt{\Delta t}(2k-N)}e^{(\mu-\frac{\sigma^2}{2})t}e^{O(\sqrt{\Delta t})}
\end{aligned} \tag{5.17}$$

Thus (5.7) becomes, ignoring the last exponential $e^{O(\sqrt{\Delta t})}$ in (5.17),

$$\begin{aligned}
\mathsf{E}^{\rm Bin}\big[\,f(S_t)\,\big] &= \sum_{k=0}^{N} f\big(S_0(1+\mu\Delta t+\sigma\sqrt{\Delta t})^k(1+\mu\Delta t-\sigma\sqrt{\Delta t})^{N-k}\big)\;\tfrac{1}{2^N}\binom{N}{k} \\
&= \sum_{k=0}^{N} f\big(S_0 e^{\sigma\sqrt{\Delta t}(2k-N)}e^{(\mu-\frac{\sigma^2}{2})t}\big)\;\tfrac{1}{2^N}\binom{N}{k} \\
&\overset{\Delta t\to 0}{\to} \int_{\mathbb{R}} f\big(S_0 e^{\sigma x}e^{(\mu-\frac{\sigma^2}{2})t}\big)\;\tfrac{1}{\sqrt{2\pi t}}\,e^{-\frac{x^2}{2t}}\,dx
\end{aligned} \tag{5.18}$$

where we used Lemma 5.1 in the last line. ■

Now that we are in a position to approximate the Black-Scholes model with a suitable Binomial model, we can consider option prices and replicating strategies. Consider first the case of a european option with payoff $H = H(S_T)$ which depends only on the stock price at maturity. According to Theorem 3.2, its theoretical fair value V_0 is given by $(t_N = T,\, t_0 = 0)$

$$V_0 \;=\; e^{-rT}\sum_{k=0}^{N} H\big(S_0(1+\mathrm{ret}_{\mathrm{up}})^k(1+\mathrm{ret}_{\mathrm{down}})^{N-k}\big) \times \tbinom{N}{k} p_{\mathrm{rn}}^k (1-p_{\mathrm{rn}})^{N-k} \tag{5.19}$$

with the risk neutral probability (3.15),

$$p_{\mathrm{rn}} \;:=\; \frac{e^{r\Delta t} - 1 - \mathrm{ret}_{\mathrm{down}}}{\mathrm{ret}_{\mathrm{up}} - \mathrm{ret}_{\mathrm{down}}} \tag{5.20}$$

and up- and down-returns given by (5.4,5.5),

$$\mathrm{ret}_{\mathrm{up}} \;=\; \mu\Delta t + \sigma\sqrt{\Delta t} \tag{5.21}$$

$$\mathrm{ret}_{\mathrm{down}} \;=\; \mu\Delta t - \sigma\sqrt{\Delta t} \tag{5.22}$$

Using $e^{r\Delta t} = 1 + r\Delta t + O\big((\Delta t)^2\big)$ and neglecting terms quadratic in Δt, we get

$$p_{\mathrm{rn}} \;=\; \frac{(r-\mu)\Delta t + \sigma\sqrt{\Delta t}}{2\sigma\sqrt{\Delta t}} \;=\; \frac{1 + \frac{r-\mu}{\sigma}\sqrt{\Delta t}}{2} \tag{5.23}$$

To obtain the option price under the Black-Scholes model, we have to calculate the $\Delta t \to 0$ limit of (5.19). A naive guess could be that the risk neutral probabilities converge actually to 1/2 and then (5.19) actually coincides with the real world expectation value (5.7) and this expression converges to the real world Black-Scholes expectation value (5.8). If this would be true, it would be actually quite bad since in that case the option price would depend on the drift parameter μ and this parameter is basically not predictable. Knowing μ is basically equivalent to knowing whether a stock is going up or down, nobody knows that. Recall that the basic result of the very elementary example in chapter 0 was that you have to buy half a stock and then you are save, regardless whether the stock is going up or down.

Fortunately this is still true in the Black-Scholes model. The $\sqrt{\Delta t}$-term in the risk neutral probabilities is actually highly important and it has the effect that in the continuous time limit the drift parameter μ completely drops out of the pricing formula, it is simply substituted by the interest rate parameter r. There is the following

Theorem 5.2: Consider the Binomial model (5.5,5.6) which converges to the Black-Scholes model (5.1). Let V_0^{Bin} be the theoretical fair value of some european option $H = H(S_T)$ in the Binomial model. Then

$$\lim_{\Delta t \to 0} V_0^{\mathrm{Bin}} \;=\; V_0^{\mathrm{BS}} \tag{5.24}$$

where the theoretical fair value under the Black-Scholes model in given by

$$V_0^{\mathrm{BS}} \;=\; e^{-rT}\int_{\mathbb{R}} H\big(S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\sqrt{T}x}\big)e^{-\frac{x^2}{2}}\,\tfrac{dx}{\sqrt{2\pi}} \tag{5.25}$$

Proof: As in the proof of Theorem 5.1, equation (5.17), we have

$$(1+\mathrm{ret_{up}})^k(1+\mathrm{ret_{down}})^{N-k} \;=\; e^{\sigma\sqrt{\Delta t}(2k-N)}e^{(\mu-\frac{\sigma^2}{2})T}e^{O(\sqrt{\Delta t})} \tag{5.26}$$

Abbreviating

$$\alpha \;:=\; \frac{r-\mu}{\sigma} \tag{5.27}$$

the risk neutral probability (5.23) is written as

$$p_{\mathrm{rn}} \;=\; \frac{1+\alpha\sqrt{\Delta t}}{2} \tag{5.28}$$

Using the Taylor expansion for $\log(1+x)$ again,

$$\begin{aligned} p_{\mathrm{rn}}^k(1-p_{\mathrm{rn}})^{N-k} &= \tfrac{1}{2^N}\, e^{k\log(1+\alpha\sqrt{\Delta t})+(N-k)\log(1-\alpha\sqrt{\Delta t})} \\ &= \tfrac{1}{2^N}\, e^{k(\alpha\sqrt{\Delta t}-\frac{\alpha^2}{2}\Delta t)+(N-k)(-\alpha\sqrt{\Delta t}-\frac{\alpha^2}{2}\Delta t)+O(\sqrt{\Delta t})} \\ &= \tfrac{1}{2^N}\, e^{-(N-2k)\alpha\sqrt{\Delta t}}e^{-\frac{\alpha^2}{2}T}e^{O(\sqrt{\Delta t})} \end{aligned} \tag{5.29}$$

Thus, the theoretical fair value in the Binomial model (5.19) becomes, again ignoring the last exponentials $e^{O(\sqrt{\Delta t})}$ in (5.28) and (5.29),

$$\begin{aligned} V_0 &= e^{-rT}\sum_{k=0}^{N} H\big(S_0(1+\mathrm{ret_{up}})^k(1+\mathrm{ret_{down}})^{N-k}\big)\,\tbinom{N}{k}p_{\mathrm{rn}}^k(1-p_{\mathrm{rn}})^{N-k} \\ &= e^{-rT}\sum_{k=0}^{N} H\big(S_0 e^{\sigma\sqrt{\Delta t}(2k-N)}e^{(\mu-\frac{\sigma^2}{2})T}\big)\,\tbinom{N}{k}\tfrac{1}{2^N}\,e^{-(N-2k)\alpha\sqrt{\Delta t}}e^{-\frac{\alpha^2}{2}T} \\ &= e^{-rT}e^{-\frac{\alpha^2}{2}T}\sum_{k=0}^{N} \underbrace{H\big(S_0 e^{\sigma\sqrt{\Delta t}(2k-N)}e^{(\mu-\frac{\sigma^2}{2})T}\big)\,e^{\alpha\sqrt{\Delta t}(2k-N)}}_{=:f[\sqrt{\Delta t}(2k-N)]\text{ for Lemma 5.1}}\;\tfrac{1}{2^N}\tbinom{N}{k} \\ &\overset{\Delta t\to 0}{\to} e^{-rT}e^{-\frac{\alpha^2}{2}T}\int_{\mathbb{R}} H\big(S_0e^{\sigma x}e^{(\mu-\frac{\sigma^2}{2})T}\big)\,e^{\alpha x}\,\tfrac{1}{\sqrt{2\pi T}}\,e^{-\frac{x^2}{2T}}dx \qquad (5.30) \\ &= e^{-rT}\int_{\mathbb{R}} H\big(S_0e^{\sigma x}e^{(\mu-\frac{\sigma^2}{2})T}\big)\,\tfrac{1}{\sqrt{2\pi T}}\,e^{-\frac{1}{2}(\frac{x}{\sqrt{T}}-\sqrt{T}\alpha)^2}dx \qquad (5.31) \end{aligned}$$

Making the substitution of variables

$$y \;=\; \frac{x}{\sqrt{T}}-\sqrt{T}\alpha \;\;\Leftrightarrow\;\; x \;=\; \sqrt{T}y+T\alpha$$

we arrive at

$$
\begin{aligned}
& e^{-rT} \int_{\mathbb{R}} H\left(S_0 e^{\sigma(\sqrt{T}y+T\alpha)} e^{(\mu-\frac{\sigma^2}{2})T}\right) \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2} dy \\
= \; & e^{-rT} \int_{\mathbb{R}} H\left(S_0 e^{\sigma\sqrt{T}y+T(r-\mu)} e^{(\mu-\frac{\sigma^2}{2})T}\right) \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2} dy \\
= \; & e^{-rT} \int_{\mathbb{R}} H\left(S_0 e^{\sigma\sqrt{T}y} e^{(r-\frac{\sigma^2}{2})T}\right) \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2} dy \qquad (5.32)
\end{aligned}
$$

which coincides with (5.25). ■