Chapter 2

The Binomial Model

Let S be some tradable asset with prices

$$S_k \quad = \quad S(t_k), \quad k = 0, 1, 2, ... \tag{2.1}$$

and let

$$H \quad = \quad H(S_0, S_1, ..., S_{N-1}, S_N) \tag{2.2}$$

be some option payoff with start date t_0 and end date or maturity t_N. We want to replicate the option payoff (2.2) with a suitable trading strategy in the underlying S. For notational simplicity let us assume first that we have zero interest rates $r = 0$. From the last chapter we know that a trading strategy holding δ_k assets at the end of day t_k generates the amount

$$V_N \quad = \quad V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{2.3}$$

Each δ_k will be determined on the end of trading day t_k. On such a day, the asset prices $S_0, S_1, ..., S_k$ are known, but the asset prices $S_{k+1}, S_{k+2}, ..., S_N$ are not known yet, they are lying in the future. Thus, δ_k can be a function only of the known prices $S_0, ..., S_k$,

$$\delta_k \quad = \quad \delta_k(S_0, S_1, ..., S_{k-1}, S_k) \tag{2.4}$$

Definition 2.1: We say that an option payoff (2.2) can be replicated by a suitable trading strategy in the underlying if and only if there are choices of δ_k of the form (2.4) and some initial amount V_0 such that (in case of zero interest rates)

$$H(S_0, S_1, ..., S_{N-1}, S_N) \quad = \quad V_0 + \sum_{j=1}^{N} \delta_{j-1}(S_j - S_{j-1}) \tag{2.5}$$

The initial amount V_0 which is needed to set up the replicating strategy is called the theoretical fair value of H.

Now let us consider the question to what extent replication of options is possible. Equation (2.5) can be rewritten as

$$H(S_0, S_1, ..., S_{N-1}, S_N) \;=\; V_{N-1} + \delta_{N-1}(S_N - S_{N-1})$$

or

$$\begin{aligned} H(S_0, S_1, ..., S_{N-1}, S_N) - \delta_{N-1}\, S_N \;&=\; V_{N-1} - \delta_{N-1}\, S_{N-1} \\ &=\; \text{some function of } \; S_0, S_1, ..., S_{N-1} \end{aligned} \tag{2.6}$$

That is, the right hand side of (2.6) is independent of S_N. Let us introduce the return of the asset S from t_{k-1} to t_k,

$$\text{ret}_k \;:=\; \frac{S_k - S_{k-1}}{S_{k-1}} \;=\; \frac{S_k}{S_{k-1}} \;-\; 1 \tag{2.7}$$

such that

$$S_k \;=\; S_{k-1}(1 + \text{ret}_k) \tag{2.8}$$

Then equation (2.6) can be rewritten as

$$H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_N)\big) - \delta_{N-1}\, S_{N-1}(1 + \text{ret}_N) \;=\; \text{const} \tag{2.9}$$

where the const in the equation above means that the left hand side of (2.9) has to be the same for all possible choices of ret_N. Since there is only 1 free parameter in (2.9), namely δ_{N-1}, we can only allow for 2 possible choices for ret_N, say,

$$\text{ret}_N \;\in \{\text{ret}_{\text{up}}, \text{ret}_{\text{down}}\} \tag{2.10}$$

and in that case we have to have

$$\begin{aligned} &H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{up}})\big) - \delta_{N-1}\, S_{N-1}(1 + \text{ret}_{\text{up}}) \;= \\ &H\big(S_0, S_1, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{down}})\big) - \delta_{N-1}\, S_{N-1}(1 + \text{ret}_{\text{down}}) \end{aligned}$$

which determines δ_{N-1} to

$$\delta_{N-1} \;=\; \frac{H\big(S_0, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{up}})\big) \;-\; H\big(S_0, ..., S_{N-1}, S_{N-1}(1 + \text{ret}_{\text{down}})\big)}{S_{N-1}(1 + \text{ret}_{\text{up}}) \;-\; S_{N-1}(1 + \text{ret}_{\text{down}})} \tag{2.11}$$

Thus, if we allow for 2 possible choices of returns only as in (2.10), replication of option payoffs seem to be possible. This leads to the following

Definition 2.2: If the price process $S_k = S(t_k)$ of some tradable asset S has the dynamics

$$S_k \quad = \quad S_{k-1}(1+\text{ret}_k) \quad \text{with} \quad \text{ret}_k \in \{\text{ret}_{\text{up}}, \text{ret}_{\text{down}}\} \; \forall \; k = 1, 2, ... \tag{2.12}$$

then we say that S is given by the Binomial model.

Remark: Observe that in Definition 2.2 we have not introduced any probabilities for an up- or down-move, that is, we have not specified a probability p_{up} such that an up-return ret_{up} will occur and a probability $p_{\text{down}} = 1-p_{\text{up}}$ for the occurence of a down-return. We did that because the replicating strategy and the theoretical option fair value V_0 are actually independent of such probabilities. Nevertheless we have to remark that the definition of the Binomial model as given in the standard literature usually includes a specification of p_{up} and $p_{\text{down}} = 1 - p_{\text{up}}$.

Now we are in a position to formulate the following important

Theorem 2.1: Let S be some tradable asset whose price process is given by the Binomial model (2.12). Let $r \geq 0$ denote some constant interest rate. Then every option payoff $H = H(S_0, ..., S_N)$ can be replicated. A replicating strategy is given by, for $k = 0, 1, ..., N-1$:

$$\delta_k \quad = \quad \frac{V_{k+1}\big(S_0, ..., S_k, S_k(1+\text{ret}_{\text{up}})\big) \; - \; V_{k+1}\big(S_0, ..., S_k, S_k(1+\text{ret}_{\text{down}})\big)}{S_k(1+\text{ret}_{\text{up}}) - S_k(1+\text{ret}_{\text{down}})} \tag{2.13}$$

and the portfolio values V_k, including the theoretical fair value V_0, can be inductively calculated through the formulae

$$V_k \quad = \quad \frac{(1 - d_{k,k+1} - d_{k,k+1}\text{ret}_{\text{down}})V_{k+1}^{\text{up}} \; - \; (1 - d_{k,k+1} - d_{k,k+1}\text{ret}_{\text{up}})V_{k+1}^{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \tag{2.14}$$

$$V_N \quad = \quad H$$

where we used the abbreviations

$$V_{k+1}^{\text{up,down}} \quad := \quad V_{k+1}\big(S_0, ..., S_k, S_k(1+\text{ret}_{\text{up,down}})\big) \tag{2.15}$$

and introduced the discount factor

$$d_{k,k+1} \quad := \quad e^{-r(t_{k+1}-t_k)} \tag{2.16}$$

Proof: For nonzero interest rates we have

$$v_N \quad = \quad v_0 + \sum_{j=1}^{N} \delta_{j-1}(s_j - s_{j-1})$$

which is equivalent to

$$v_{k+1} \;=\; v_k + \delta_k (s_{k+1} - s_k) \quad \forall\, k = 0, 1, ..., N-1$$

or, since $v_k = e^{-r(t_k - t_0)} V_k$,

$$e^{-r(t_{k+1}-t_k)} V_{k+1} \;=\; V_k + \delta_k \left(e^{-r(t_{k+1}-t_k)} S_k (1 + \text{ret}_k) - S_k \right)$$

Since we assume the price dynamics of the Binomial model, this equation is equivalent to

$$e^{-r(t_{k+1}-t_k)} V_{k+1}^{\text{up}} \;=\; V_k + \delta_k \left(e^{-r(t_{k+1}-t_k)} S_k (1 + \text{ret}_{\text{up}}) - S_k \right) \tag{2.17}$$

$$e^{-r(t_{k+1}-t_k)} V_{k+1}^{\text{down}} \;=\; V_k + \delta_k \left(e^{-r(t_{k+1}-t_k)} S_k (1 + \text{ret}_{\text{down}}) - S_k \right) \tag{2.18}$$

Subtracting (2.18) from (2.17) gives

$$e^{-r(t_{k+1}-t_k)} (V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}) \;=\; \delta_k \left(e^{-r(t_{k+1}-t_k)} S_k (1 + \text{ret}_{\text{up}}) - e^{-r(t_{k+1}-t_k)} S_k (1 + \text{ret}_{\text{down}}) \right)$$

or

$$\delta_k \;=\; \frac{V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}}{S_k(1 + \text{ret}_{\text{up}}) - S_k(1 + \text{ret}_{\text{down}})} \tag{2.19}$$

which coincides with (2.13). Substituting this value of δ_k in equation (2.17) and solving for V_k gives (now using the abbreviation (2.16) for the discount factors)

$$d_{k,k+1} V_{k+1}^{\text{up}} - \delta_k S_k \big(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1 \big) \;=\; V_k \quad \Leftrightarrow$$

$$\begin{aligned}
V_k \;&=\; d_{k,k+1} V_{k+1}^{\text{up}} - \frac{V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \big(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1 \big) \\
&=\; \frac{d_{k,k+1} V_{k+1}^{\text{up}} (\text{ret}_{\text{up}} - \text{ret}_{\text{down}}) - V_{k+1}^{\text{up}} \big(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1 \big) + V_{k+1}^{\text{down}} \big(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1 \big)}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \\
&=\; \frac{-V_{k+1}^{\text{up}} \big(d_{k,k+1}(1 + \text{ret}_{\text{down}}) - 1 \big) + V_{k+1}^{\text{down}} \big(d_{k,k+1}(1 + \text{ret}_{\text{up}}) - 1 \big)}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \\
&=\; \frac{V_{k+1}^{\text{up}} \big(1 - d_{k,k+1}(1 + \text{ret}_{\text{down}}) \big) - V_{k+1}^{\text{down}} \big(1 - d_{k,k+1}(1 + \text{ret}_{\text{up}}) \big)}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}}
\end{aligned}$$

and this coincides with (2.14). ■

Remarks: 1) If H is some (then usually called "european") option which depends only on the underlying price at maturity,

$$H \;=\; H(S_N) \tag{2.20}$$

then the δ_k and the value of the replicating portfolio V_k at t_k depend only on the asset price S_k and do not depend on earlier prices $S_{k-1}, S_{k-2}, ..., S_0$. That is,

$$V_k = V_k(S_k) \tag{2.21}$$
$$\delta_k = \delta_k(S_k) \tag{2.22}$$

2) Assume zero interest rates such that $d_{k,k+1} = 1$. Then (2.14) becomes

$$\begin{aligned} V_k &= \frac{-\text{ret}_{\text{down}} V_{k+1}^{\text{up}} + \text{ret}_{\text{up}} V_{k+1}^{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \\ &= \frac{V_{k+1}^{\text{up}} + V_{k+1}^{\text{down}}}{2} - \frac{\text{ret}_{\text{up}} + \text{ret}_{\text{down}}}{2} \frac{V_{k+1}^{\text{up}} - V_{k+1}^{\text{down}}}{\text{ret}_{\text{up}} - \text{ret}_{\text{down}}} \end{aligned} \tag{2.23}$$

If we further assume a "symmetric" Binomial model with $\text{ret}_{\text{up}} = q\%$ and $\text{ret}_{\text{down}} = -q\%$ we obtain the simple recursion formula

$$V_k = \frac{V_{k+1}^{\text{up}} + V_{k+1}^{\text{down}}}{2} \tag{2.24}$$