Chapter 10

Probabilities Involving the Minimum and the Maximum of a Brownian Motion

Probabilities involving the minimum or maximum of a Brownian motion show up in the valuation of barrier and lookback options. These are european options whose payoff H depends on S_T, the stock price at maturity T, and on $M_T := \max_{0\leq t\leq T} S_t$ or $m_T := \min_{0\leq t\leq T} S_t$,

$$H = H\big(S_T, \min_{0\leq t\leq T} S_t\big) \tag{10.1}$$

Thus these options depend on the particular path which leads to S_T, but only in a weak sense. For the Binomial model and the Black-Scholes model there are explicit pricing formulae. In this and the next section we derive these formulae for the Black-Scholes model. The corresponding results in the Binomial model can be found, for example, in [4].

Let $\{x_t\}_{0\leq t\leq T}$ be a Brownian motion and let

$$S_t^{(\mu)} = S_0\, e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t} \tag{10.2}$$

be the price process. In the last chapter we saw that the price V_0 of an arbitrary european option is given by the expectation with respect to the equivalent martingale measure,

$$\begin{aligned} V_0 &= e^{-rT}\mathsf{E}_{\tilde W}\Big[H\big(S_T^{(\mu)}, \min_{0\leq t\leq T} S_t^{(\mu)}\big)\Big] \\ &= e^{-rT}\mathsf{E}_{W}\Big[H\big(S_T^{(r)}, \min_{0\leq t\leq T} S_t^{(r)}\big)\Big] \end{aligned} \tag{10.3}$$

where we used Theorem 9.2 in the second line. Because of

$$\min_{0\leq t\leq T} S_t^{(r)} = S_0\, e^{\sigma \min_{0\leq t\leq T}\left\{x_t + (\frac{r}{\sigma} - \frac{\sigma}{2})t\right\}} \tag{10.4}$$

we may equally well consider H as a function of x_T and $\min_{0\leq t\leq T}\left\{x_t+(\frac{r}{\sigma}-\frac{\sigma}{2})t\right\}$. We first make a substitution of variables to eliminate the $(\frac{r}{\sigma}-\frac{\sigma}{2})t$ term in the minimum. There is the following

Lemma 10.1 (Girsanov): Let $dW(\{x_t\}_{0<t\leq T})$ be the Wiener measure and make the substitution of variables $y_t = x_t + ct$. Then

$$dW(\{y_t\}_{0<t\leq T}) \;=\; dW(\{x_t+ct\}_{0<t\leq T}) \;=\; e^{-cx_T-\frac{c^2}{2}T}\, dW(\{x_t\}_{0<t\leq T}) \tag{10.5}$$

Proof: Let $t_k = k\Delta t$ and $N_T = T/\Delta t$. Then

$$\begin{aligned} dW(\{y_t\}_{0<t\leq T}) &= \lim_{\Delta t\to 0}\prod_{k=1}^{N_T} p_{\Delta t}(y_{t_{k-1}},y_{t_k})\,dy_{t_k} \\ &= \lim_{\Delta t\to 0}\prod_{k=1}^{N_T}\frac{1}{\sqrt{2\pi\Delta t}}\,e^{-\frac{(y_{t_k}-y_{t_{k-1}})^2}{2\Delta t}}\,dy_{t_k} \\ &= \lim_{\Delta t\to 0}\prod_{k=1}^{N_T}\frac{1}{\sqrt{2\pi\Delta t}}\,e^{-\frac{\left[x_{t_k}-x_{t_{k-1}}+c(t_k-t_{k-1})\right]^2}{2\Delta t}}\,dx_{t_k} \\ &= \lim_{\Delta t\to 0}\prod_{k=1}^{N_T}\frac{1}{\sqrt{2\pi\Delta t}}\,e^{-\frac{(x_{t_k}-x_{t_{k-1}})^2}{2\Delta t}}\,e^{-(x_{t_k}-x_{t_{k-1}})c-\frac{c^2}{2}\Delta t}\,dx_{t_k} \\ &= \lim_{\Delta t\to 0}\prod_{k=1}^{N_T}\frac{1}{\sqrt{2\pi\Delta t}}\,e^{-\frac{(x_{t_k}-x_{t_{k-1}})^2}{2\Delta t}}\,dx_{t_k}\,e^{-\sum_{k=1}^{N_T}(x_{t_k}-x_{t_{k-1}})c-\frac{c^2}{2}N_T\Delta t} \\ &= dW(\{x_t\}_{0<t\leq T})\,e^{-(x_T-x_0)c-\frac{c^2}{2}T} \\ &= e^{-cx_T-\frac{c^2}{2}T}\,dW(\{x_t\}_{0<t\leq T}) \end{aligned} \tag{10.6}$$

which proves the lemma. ■

Using the above lemma with $c=\frac{r}{\sigma}-\frac{\sigma}{2}$, (10.3) becomes

$$\begin{aligned} v_0 &= e^{-rT}\int H\left(S_0\,e^{\sigma(x_T+cT)},S_0\,e^{\sigma\min_{0\leq t\leq T}\{x_t+ct\}}\right)dW(\{x_t\}_{0<t\leq T}) \\ &= e^{-rT}\int H\left(S_0\,e^{\sigma y_T},S_0\,e^{\sigma\min_{0\leq t\leq T}\{y_t\}}\right)e^{cx_T+\frac{c^2}{2}T}dW(\{y_t\}_{0<t\leq T}) \\ &= e^{-rT}\int H\left(S_0\,e^{\sigma y_T},S_0\,e^{\sigma\min_{0\leq t\leq T}\{y_t\}}\right)e^{cy_T-\frac{c^2}{2}T}dW(\{y_t\}_{0<t\leq T}) \end{aligned} \tag{10.7}$$

Thus, we have to compute an expectation value which depends on

$$F(\{y_t\}) \;:=\; y_T \tag{10.8}$$

$$G(\{y_t\}) \;:=\; \min_{0\leq t\leq T}\{\,y_t\,\} \tag{10.9}$$

Let I be the integrand in (10.7), $I = H(\cdots)\, e^{cy_T - \frac{c^2}{2}T} = I(F,G)$. For real valued F, let $1 = \sum_a \chi\big(F \in [a, a+\Delta a)\big)$ be a partition of the real axis. We write

$$\begin{aligned}
&\mathsf{E}_W\Big[I\big(F(\{x_t\}), G(\{x_t\})\big)\Big] \\
&= \lim_{\substack{\Delta a\to 0\\ \Delta b\to 0}} \sum_{a,b} \mathsf{E}_W\Big[I\big(F(\{x_t\}), G(\{x_t\})\big)\chi\big(F\in[a,a+\Delta a)\big)\chi\big(G\in[b,b+\Delta b)\big)\Big] \\
&= \lim_{\substack{\Delta a\to 0\\ \Delta b\to 0}} \sum_{a,b} I(a,b)\, \mathsf{E}_W\Big[\chi\big(F\in[a,a+\Delta a)\big)\chi\big(G\in[b,b+\Delta b)\big)\Big] \\
&= \lim_{\substack{\Delta a\to 0\\ \Delta b\to 0}} \sum_{a,b} I(a,b)\, \mathsf{P}_W\big(F\in[a,a+\Delta a),\ G\in[b,b+\Delta b)\big) \\
&= \int_{\mathbb{R}^2} I(a,b)\, \rho_{F,G}(a,b)\, da\, db
\end{aligned} \tag{10.10}$$

where we introduced the joint distribution

$$\mathsf{P}_W\big(F\in[a,a+da),\ G\in[b,b+db)\big) \;=\; \rho_{F,G}(a,b)\, da\, db \tag{10.11}$$

For $F(\{y_t\}) = y_T$ and $G(\{y_t\}) = \min_{0\le t\le T}\{\, y_t \,\}$, the joint distribution can be calculated explicitely. This is a consequence of the reflection principle which is formulated in the next lemma. For the statement, we need the definition of a stopping time.

Definition 10.2: A function

$$\tau : \{x_t\}_{0\le t\le T} \to \tau\big(\{x_t\}_{0\le t\le T}\big) \in \mathbb{R} \tag{10.12}$$

is called a stopping time, if the value of τ depends only on $\{x_t\}_{0\le t\le \tau}$,

$$\tau\big(\{x_t\}_{0\le t\le\tau}, \{y_t\}_{\tau<t\le T}\big) \;=\; \tau\big(\{x_t\}_{0\le t\le\tau}, \{y'_t\}_{\tau<t\le T}\big) \quad \forall y_t, y'_t \tag{10.13}$$

or in a more compact notation $\tau = \tau\big(\{x_t\}_{0\le t\le\tau}\big)$.

Example 10.3: Let $b \in \mathbb{R}$. Then the assignment

$$\tau_b(\{x_t\}) := \inf_{0\le s\le T}\{s \,|\, x_s = b\} \tag{10.14}$$

defines a stopping time, but $\sigma_b(\{x_t\}) := \sup_{0\le s\le T}\{s\,|\,x_s = b\}$ is not a stopping time.

Lemma 10.4 (Reflection Principle): Let $dW(\{x_t\}_{0<t\le T})$ be the Wiener measure and let τ be a stopping time. For a path x_t, define the at the level x_τ reflected path y_t by

$$y_t := \begin{cases} x_t & \text{if } t \le \tau(\{x_t\}) \\ 2x_\tau - x_t & \text{if } t > \tau(\{x_t\}) \end{cases} \tag{10.15}$$

Then

$$dW(\{x_t\}_{0<t\leq T}) \quad = \quad dW(\{y_t\}_{0<t\leq T}) \tag{10.16}$$

Proof: Observe first that the transformation (10.15) is invertible, the inverse transformation being identical to the original one. Geometrically, this is quite obvious. More formally, since $\tau(\{x_t\}) = \tau(\{x_t\}_{0\leq t\leq \tau})$ and because of $y_t = x_t$ for $t \leq \tau$, $\tau = \tau(\{y_t\}_{0\leq t\leq \tau})$. For $t > \tau$, $x_t = 2x_\tau - y_t = 2y_\tau - y_t$ which is identical to the transformation in (10.15).

We approximate the Wiener measure by its finite dimensional distributions. For simplicity, consider only those Δt such that $N_\tau = \tau/\Delta t \in \mathbb{N}$. Then

$$\begin{aligned}
& dW(\{x_t\}_{0<t\leq T}) \\
&= \lim_{\Delta t\to 0} \prod_{k=1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t})\, dx_{k\Delta t} \\
&= \lim_{\Delta t\to 0} \prod_{k=1}^{N_\tau} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t}) \prod_{k=N_\tau+1}^{N_T} p_{\Delta t}(x_{(k-1)\Delta t}, x_{k\Delta t}) \prod_{k=1}^{N_T} dx_{k\Delta t} \\
&= \lim_{\Delta t\to 0} \prod_{k=1}^{N_\tau} p_{\Delta t}(y_{(k-1)\Delta t}, y_{k\Delta t})\, p_{\Delta t}(y_{N_\tau \Delta t}, 2y_\tau - y_{N_\tau\Delta t+\Delta t}) \times \\
&\qquad \prod_{k=N_\tau+2}^{N_T} p_{\Delta t}(2y_\tau - y_{(k-1)\Delta t}, 2y_\tau - y_{k\Delta t}) \left|\det\left(\tfrac{\partial x}{\partial y}\right)\right| \prod_{k=1}^{N_T} dy_{k\Delta t} \\
&= \lim_{\Delta t\to 0} \prod_{k=1}^{N_\tau} p_{\Delta t}(y_{(k-1)\Delta t}, y_{k\Delta t})\, p_{\Delta t}(y_{N_\tau \Delta t}, y_{N_\tau\Delta t+\Delta t}) \prod_{k=N_\tau+2}^{N_T} p_{\Delta t}(y_{(k-1)\Delta t}, y_{k\Delta t}) \prod_{k=1}^{N_T} dy_{k\Delta t} \\
&= dW(\{y_t\}_{0<t\leq T})
\end{aligned} \tag{10.17}$$

since $[y_\tau - (2y_\tau - y_{\tau+\Delta t})]^2 = [y_\tau - y_{\tau+\Delta t}]^2$, $[2y_\tau - y_{(k-1)\Delta t} - (2y_\tau - y_{k\Delta t})]^2 = [y_{(k-1)\Delta t} - y_{k\Delta t}]^2$ and

$$\det\left(\frac{\partial x}{\partial y}\right) \quad = \quad \det \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & 2 & -1 & \\ & & \vdots & & \ddots & \\ & & 2 & & & -1 \end{pmatrix} \quad = \quad \pm 1 \tag{10.18}$$

In the matrix above, the column with the 2's is column number N_τ. ■

Theorem 10.5: Let dW be the Wiener measure, let $\{x_t\}_{0\leq t\leq T}$ be a Brownian motion and recall

$$N(x) = \int_{-\infty}^{x} \varphi(y)\, dy, \qquad \varphi(y) = \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}} \tag{10.19}$$

Let a, b be positive real numbers. Then we have the following probabilities:

a)

$$\mathsf{P}_W\big(\max_{0\le t\le T} x_t \ge a,\ x_T < a-b\big) \;=\; \mathsf{P}_W\big(x_T > a+b\big) \tag{10.20}$$

$$\mathsf{P}_W\big(\max_{0\le t\le T} x_t \ge a\big) \;=\; 2\,\mathsf{P}_W\big(x_T \ge a\big) \tag{10.21}$$

b)

$$\begin{aligned}\mathsf{P}_W\big(x_T \le a,\ \max_{0\le t\le T} x_t \le b\big) \;&=\; \mathsf{P}_W\big(x_T \ge -a,\ \min_{0\le t\le T} x_t \ge -b\big)\\ &=\; \begin{cases} N\big(a/\sqrt{T}\big) + N\big((2b-a)/\sqrt{T}\big) - 1 & \text{if } a \le b \\ 2N\big(b/\sqrt{T}\big) - 1 & \text{if } a > b\end{cases}\end{aligned} \tag{10.22}$$

c) In particular, there is the density

$$\begin{aligned}&\mathsf{P}_W\Big(x_T \in [a, a+da),\ \max_{0\le t\le T} x_t \in [b, b+db)\Big) \;= \\ &\mathsf{P}_W\Big(x_T \in [-a, -a+da),\ \min_{0\le t\le T} x_t \in [-b, -b+db)\Big) \;=\; -\tfrac{2}{T}\,\varphi'\big(\tfrac{2b-a}{\sqrt{T}}\big)\,\chi(a<b)\end{aligned} \tag{10.23}$$

Proof: a) Define the stopping time

$$\tau_a(\{x_t\}) = \inf_{t\in(0,\infty)}\{t\,|\,x_t = a\} \tag{10.24}$$

Then

$$\big\{x_t\,|\,\max_{0\le t\le T} x_t \ge a\big\} \;=\; \big\{x_t\,|\,\tau_a(\{x_t\}) \le T\big\} \tag{10.25}$$

For given x_t, define the at the level $x_{\tau_a} = a$ reflected path as in (10.15), $y_t = x_t$ for $t \le \tau_a$ and $y_t = 2x_{\tau_a} - x_t = 2a - x_t$ for $t > \tau_a$. In particular, $\tau_a = \tau_a(\{x_t\}_{0\le t\le\tau_a}) = \tau_a(\{y_t\}_{0\le t\le\tau_a})$. Then

$$\begin{aligned}\mathsf{P}_W\big(\max_{0\le t\le T} x_t \ge a,\ x_T < a-b\big) \;&=\; \mathsf{P}_W\big(\tau_a \le T,\ x_T < a-b\big)\\ &=\; \int \chi\big(\tau_a(\{x_t\}_{0\le t\le\tau_a}) \le T,\ x_T < a-b\big)\,dW(\{x_t\})\\ &=\; \int \chi\big(\tau_a(\{y_t\}_{0\le t\le\tau_a}) \le T,\ 2a - y_T < a-b\big)\,dW(\{y_t\})\\ &=\; \int \chi\big(\tau_a(\{y_t\}_{0\le t\le\tau_a}) \le T,\ y_T > a+b\big)\,dW(\{y_t\})\\ &=\; \int \chi\big(y_T > a+b\big)\,dW(\{y_t\})\\ &=\; \mathsf{P}_W\big(x_T > a+b\big)\end{aligned} \tag{10.26}$$

In the third line of (10.26) we used the reflection principle Lemma 10.4 whereas in the last line we simply renamed the integration variables from y_t to x_t. In particular, for $b = 0$,

$$\mathsf{P}_W\big(\max_{0\le t\le T} x_t \ge a,\ x_T < a\big) \;=\; \mathsf{P}_W\big(x_T > a\big) \tag{10.27}$$

Using this, (10.21) follows from

$$\begin{aligned}
\mathsf{P}_W\big(\max_{0\le t\le T} x_t \ge a\big) &= \mathsf{P}_W\big(\max_{0\le t\le T} x_t \ge a, x_T < a\big) + \mathsf{P}_W\big(\max_{0\le t\le T} x_t \ge a, x_T \ge a\big) \\
&\overset{(10.27)}{=} \mathsf{P}_W\big(x_T > a\big) + \mathsf{P}_W\big(x_T \ge a\big) \\
&= 2\,\mathsf{P}_W\big(x_T \ge a\big)
\end{aligned} \tag{10.28}$$

b) For $b < a$ we have

$$\begin{aligned}
\mathsf{P}_W\big(x_T \le a,\ \max_{0\le t\le T} x_t \le b\big) &= \mathsf{P}_W\big(\max_{0\le t\le T} x_t \le b\big) \\
&= 1 - \mathsf{P}_W\big(\max_{0\le t\le T} x_t > b\big) \\
&\overset{(a)}{=} 1 - 2\,\mathsf{P}_W\big(x_T > b\big) \\
&= 2\,\mathsf{P}_W\big(x_T \le b\big) - 1 \\
&= 2\int_{\mathbb{R}} \chi(x_T \le b)\, p_T(0, x_T)\, dx_T \;-\; 1 \\
&= 2\int_{-\infty}^{b/\sqrt{T}} \tfrac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy \;-\; 1 \\
&= 2N\big(b/\sqrt{T}\big) - 1
\end{aligned} \tag{10.29}$$

For $a \le b$,

$$\begin{aligned}
\mathsf{P}_W\big(x_T \le a,\ \max_{0\le t\le T} x_t \le b\big) &= \mathsf{P}_W\big(x_T \le a,\big) - \mathsf{P}_W\big(x_T \le a,\ \max_{0\le t\le T} x_t > b\big) \\
&= \mathsf{P}_W\big(x_T \le a,\big) - \mathsf{P}_W\big(\max_{0\le t\le T} x_t > b,\ x_T \le b - (b-a)\big) \\
&\overset{(a)}{=} \mathsf{P}_W\big(x_T \le a,\big) - \mathsf{P}_W\big(x_T > b + (b-a)\big) \\
&= \mathsf{P}_W\big(x_T \le a,\big) + \mathsf{P}_W\big(x_T \le 2b - a\big) - 1 \\
&= N\big(a/\sqrt{T}\big) + N\big((2b-a)/\sqrt{T}\big) - 1
\end{aligned} \tag{10.30}$$

This proves (10.22). The density in (10.23) in part (c), say $\rho(a,b)$, is obtained by differentiation,

$$\rho(a,b) \;=\; \frac{\partial}{\partial a}\frac{\partial}{\partial b}\mathsf{P}_W\big(\, x_T \le a,\ \max_{0\le t\le T} x_t \le b\,\big) \tag{10.31}$$

One computes

$$\begin{aligned}\rho(a,b) \;&=\; \frac{\partial}{\partial b}\begin{cases} \frac{1}{\sqrt{T}}\Big(\varphi\big(\frac{a}{\sqrt{T}}\big) - \varphi\big(\frac{2b-a}{\sqrt{T}}\big)\Big) & \text{if } a<b \\ 0 & \text{if } a>b \end{cases} \\ &=\; -\tfrac{2}{T}\,\varphi'\big(\tfrac{2b-a}{\sqrt{T}}\big)\,\chi(a<b) \end{aligned} \tag{10.32}$$

Since the first line of (10.32) is continuous at $a=b$, there is no δ-function at $a=b$. Finally observe that

$$\begin{aligned}\mathsf{P}_W\big(\, x_T \ge -a,\ \min_{0\le t\le T} x_t \ge -b\,\big) \;&=\; \mathsf{P}_W\big(\, -x_T \le a,\ -\min_{0\le t\le T} x_t \le b\,\big) \\ &=\; \mathsf{P}_W\big(\, -x_T \le a,\ \max_{0\le t\le T}\{-x_t\} \le b\,\big) \\ &=\; \mathsf{P}_W\big(\, x_T \le a,\ \max_{0\le t\le T} x_t \le b\,\big)\end{aligned}$$

since $dW(\{-x_t\}) = dW(\{x_t\})$. ■