Chapter 15

Ito-Diffusions and the Ornstein-Uhlenbeck Process

Ito Diffusions

A d-dimensional Ito process $X_t = (X_t^1, ..., X_t^d)$ is a stochastic process of the form $X_t = X_0 + \int_0^t u\,ds + \int_0^t v\,dB_s$ or, in differential form,

$$dX_t = u\big(t, \{B_s\}_{0\le s\le t}\big)\,dt + v\big(t, \{B_s\}_{0\le s\le t}\big)\,dB_t \tag{15.1}$$

The function u takes values in $\mathbb{R}^d$, v takes values in $\mathbb{R}^{d\times m}$ and B_t is an m-dimensional (uncorrelated) Brownian motion. The functions u and v are non anticipating. That is, at time t, u and $B_{t+h} - B_t$ are statistically independent for all $h > 0$. Therefore $u = u\big(t, \{B_s\}_{0\le s\le t}\big)$. If $f = f(t, x_1, ..., x_d)$ is some $\mathbb{R}$ valued C^2 function, then, by the Ito formula, $Y_t := f(t, X_t)$ is again a (one dimensional) Ito process with differential

$$dY_t \;=\; \frac{\partial f}{\partial t}(t, X_t)\,dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(t, X_t)\,dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x_i \partial x_j}(t, X_t)\,dX_t^i dX_t^j \tag{15.2}$$

where $dX_t^i dX_t^j$ has to be computed according to the rules $dt\cdot dt = dt\cdot dB_t^i = dB_t^i\cdot dt = 0$ and $dB_t^i \cdot dB_t^j = \delta_{ij}\,dt$.

A d-dimensional Ito diffusion $X_t = (X_t^1, ..., X_t^d)$ is a solution to the stochastic differential equation (SDE)

$$dX_t \;=\; \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t \tag{15.3}$$

where μ takes values in $\mathbb{R}^d$, σ takes values in $\mathbb{R}^{d\times m}$ and B_t is an m-dimensional (uncorrelated) Brownian motion. Stochastic differential equations are used to model the behaviour of many financial assets. The proof of the following existence and uniqueness result can be found, for example, in the book of Oksendal [12].

Theorem 15.1: For a given SDE (15.3) with some initial condidtion $X_0 = x_0$, there exists a unique t-continuous solution X_t in $[0, T]$ if the following two conditions are fulfilled:

(i) Lipschitz condition:

$$|\mu(t,x)-\mu(t,y)|+|\sigma(t,x)-\sigma(t,y)| \;\le\; D\,|x-y|\,, \qquad x,y\in\mathbb{R}^d,\;\; t\in[0,T] \tag{15.4}$$

(ii) growth condition:

$$|\mu(t,x)|+|\sigma(t,x)| \;\le\; C\,(1+|x|)\,, \qquad x\in\mathbb{R}^d,\;\; t\in[0,T] \tag{15.5}$$

Here $|\sigma|^2=\sum_{i,j}|\sigma_{ij}|^2$. The solution is non anticipating,

$$X_t \;=\; X\big(t,\{B_s\}_{0\le s\le t}\big) \tag{15.6}$$

Sketch of Proof: The proof is similar to the proof for the deterministic case. The solution is obtained by iteration. Let us first prove the following inequality (15.10) from which existence and uniqueness follows. Suppose X_t and $\hat{X}_t$ are solutions of (15.3). That is,

$$X_t \;=\; X_0+\int_0^t\mu(s,X_s)\,ds+\int_0^t\sigma(s,X_s)\,dB_s \tag{15.7}$$

and the same for $\hat{X}_t$ with initial condition, say, $\hat{X}_0$. Let

$$a_s \;:=\; \mu(s,X_s)-\mu(s,\hat{X}_s) \tag{15.8}$$

$$\gamma_s \;:=\; \sigma(s,X_s)-\sigma(s,\hat{X}_s) \tag{15.9}$$

Then

$$\begin{aligned}
\mathsf{E}\big[\,|X_t-\hat{X}_t|^2\big] &= \mathsf{E}\Big[\Big(X_0-\hat{X}_0+\int_0^t a_s\,ds+\int_0^t\gamma_s\,dB_s\Big)^2\Big]\\
&\le 3\mathsf{E}\big[\,|X_0-\hat{X}_0|^2\big]+3\mathsf{E}\Big[\Big(\int_0^t a_s\,ds\Big)^2\Big]+3\mathsf{E}\Big[\Big(\int_0^t\gamma_s\,dB_s\Big)^2\Big]\\
&\le 3\mathsf{E}\big[\,|X_0-\hat{X}_0|^2\big]+3t\mathsf{E}\Big[\int_0^t a_s^2\,ds\Big]+3\mathsf{E}\Big[\int_0^t\gamma_s^2\,ds\Big]\\
&\le 3\mathsf{E}\big[\,|X_0-\hat{X}_0|^2\big]+3(1+t)D^2\mathsf{E}\Big[\int_0^t|X_s-\hat{X}_s|^2ds\Big]\\
&= 3\mathsf{E}\big[\,|X_0-\hat{X}_0|^2\big]+3(1+t)D^2\int_0^t\mathsf{E}\big[\,|X_s-\hat{X}_s|^2\big]ds
\end{aligned} \tag{15.10}$$

Thus, for $t\in[0,T]$, the function

$$v(t) \;=\; \mathsf{E}\big[\,|X_t-\hat{X}_t|^2\big] \tag{15.11}$$

satisfies

$$v(t) \;\le\; F+A\int_0^t v(s)\,ds \tag{15.12}$$

if we put $F = 3\mathsf{E}\big[\,|X_0 - \hat{X}_0|^2\big]$ and $A = 3(1+T)D^2$. From this, using the Gronwall lemma, one deduces that

$$v(t) \;\leq\; Fe^{At} \tag{15.13}$$

Thus uniqueness follows if the initial conditions are the same, $X_0 = \hat{X}_0$. To construct a solution, define $Y_t^{(0)} := X_0$ and iterate the equation

$$Y_t^{(k+1)} \;:=\; X_0 + \int_0^t \mu(s, Y_s^{(k)})\, ds + \int_0^t \sigma(s, Y_s^{(k)})\, dB_s \tag{15.14}$$

By construction, all the $Y_t^{(k)}$ are non anticipating, $Y_t^{(k)} = Y_t^{(k)}\big(\{B_s\}_{0\leq s\leq t}\big)$. From inequality (15.10) we get

$$\mathsf{E}\big[\,|Y_t^{(k+1)} - Y_t^{(k)}|^2\big] \;\leq\; 3(1+T)D^2 \int_0^t \mathsf{E}\big[\,|Y_t^{(k)} - Y_t^{(k-1)}|^2\big] ds \tag{15.15}$$

for $k \geq 1$ and $\mathsf{E}\big[\,|Y_t^{(1)} - Y_t^{(0)}|^2\big] \leq A_1\, t$ where A_1 depends only on C, T and $\mathsf{E}[X_0^2]$. By induction, one obtains

$$\mathsf{E}\big[\,|Y_t^{(k+1)} - Y_t^{(k)}|^2\big] \;\leq\; \tfrac{(A_2 t)^{k+1}}{(k+1)!} \quad k \geq 0,\ t \in [0,T] \tag{15.16}$$

with some suitable constant A_2 depending only on C, D, T and $\mathsf{E}[X_0^2]$. From this one can deduce that the sequence

$$Y_t^{(n)}\big(\{B_s\}\big) = Y_t^{(0)}\big(\{B_s\}\big) + \sum_{k=0}^{n-1} \Big(Y_t^{(k+1)}\big(\{B_s\}\big) - Y_t^{(k)}\big(\{B_s\}\big)\Big) \tag{15.17}$$

is uniformly convergent in $[0,T]$ for almost all Brownian paths $\{B_s\}_{0\leq s\leq t}$. ∎

From the above uniqueness result follows the Markov property for the solution of (15.3). There is the following

Theorem 15.2: Let $X_t = X_t^{t_0,x_0}$ be the unique solution of (15.3) with initial condition $X_{t_0} = x_0 \in \mathbb{R}^d$. Let $t_0 < t_1 < \cdots < t_n$ be some times and $x_1, ..., x_n \in \mathbb{R}^d$. Define the probability densities

$$\mathsf{P}_W\Big(X_{t_1} \in [x_1, x_1 + dx_1), ..., X_{t_n} \in [x_n, x_n + dx_n)\Big) \;=:\; p(t_0,x_0; t_1,x_1; ...; t_n,x_n)\, dx_1 \cdots dx_n \tag{15.18}$$

where P_W is the probability with respect to the Wiener measure. Then

$$p(t_0,x_0; t_1,x_1; ...; t_n,x_n) \;=\; p(t_0,x_0; t_1,x_1)\, p(t_1,x_1; t_2,x_2) \cdots p(t_{n-1},x_{n-1}; t_n,x_n) \tag{15.19}$$

where for $s < t$

$$p(s, x\,;\, t, y)\, dy \;=\; \mathsf{P}_W\big(X_t^{s,x} \in [y, y + dy)\big) \tag{15.20}$$

In particular, for some function $F : \mathbb{R}^n \to \mathbb{R}$,

$$\mathsf{E}_W\big[F(X_{t_1}, ..., X_{t_n})\big] \;=\; \int_{(\mathbb{R}^d)^n} F(x_1, ..., x_n) \prod_{j=1}^{n} p(t_{j-1}, x_{j-1}\,;\, t_j, x_j)\, dx_j \tag{15.21}$$

Proof: Let $dW_{(t_0,T]}^{y_0}$ be the Wiener measure on paths on $(t_0, T]$, $T > t_n$, which start at y_0 ($t_j = j\Delta t$, $N_T = T/\Delta t$),

$$\begin{aligned} dW_{(t_0,T]}^{y_0}\big(\{y_t\}_{t_0<t\le T}\big) \;&=\; \lim_{\Delta t\to 0} \prod_{j=N_{t_0}+1}^{N_T} p_{\Delta t}(y_{t_{j-1}}, y_{t_j})\, dy_{t_j} \\ &=\; dW_{(t_0,t_1]}^{y_0}\big(\{y_t\}_{t_0<t\le t_1}\big)\, dW_{(t_1,t_2]}^{y_{t_1}}\big(\{y_t\}_{t_1<t\le t_2}\big) \cdots dW_{(t_n,T]}^{y_{t_n}}\big(\{y_t\}_{t_n<t\le T}\big) \end{aligned} \tag{15.22}$$

Since the solution of (15.3) can be obtained by iteration of the equation $X_t = x_0 + \int_{t_0}^t \mu(s, X_s)\, ds + \int_{t_0}^t \sigma(s, X_s)\, dB_s$, we have that

$$X_t^{t_0,x_0} = X_t\big(\{B_s\}_{t_0<s\le t}\big) \tag{15.23}$$

The probability in (15.18) can be written as

$$\begin{aligned} &\mathsf{P}_W\Big(X_{t_1} \in [x_1, x_1 + dx_1), ..., X_{t_n} \in [x_n, x_n + dx_n)\Big) \\ &\quad= \int \chi\Big(X_{t_1} \in [x_1, x_1 + dx_1), ..., X_{t_n} \in [x_n, x_n + dx_n)\Big) dW_{(t_0,T]}^{y_0} \\ &\quad\overset{(15.23)}{=} \int \chi\big(X_{t_1} \in [x_1, x_1 + dx_1)\big) \cdots \chi\big(X_{t_n} \in [x_n, x_n + dx_n)\big) dW_{(t_0,t_n]}^{y_0} \\ &\quad\overset{(15.22)}{=} \int \chi\big(X_{t_1} \in [x_1, x_1 + dx_1)\big) \cdots \chi\big(X_{t_n} \in [x_n, x_n + dx_n)\big) \prod_{j=1}^{n} dW_{(t_{j-1},t_j]}^{y_{t_{j-1}}}\big(\{y_t\}_{t_{j-1}<t\le t_j}\big) \end{aligned} \tag{15.24}$$

By the uniqueness result, we have, for $t > t_j$,

$$X_t^{t_0,x_0} = X_t^{t_j, X_{t_j}^{t_0,x_0}} \tag{15.25}$$

such that

$$\begin{aligned} &\chi\big(X_{t_1} \in [x_1, x_1 + dx_1)\big) \cdots \chi\big(X_{t_n} \in [x_n, x_n + dx_n)\big) \\ &\quad= \chi\big(X_{t_1}^{t_0,x_0} \in [x_1, x_1 + dx_1)\big)\, \chi\big(X_{t_2}^{t_1,X_{t_1}} \in [x_2, x_2 + dx_2)\big) \times \cdots \\ &\qquad\qquad \cdots \times \chi\big(X_{t_n}^{t_{n-1},X_{t_{n-1}}} \in [x_n, x_n + dx_n)\big) \\ &\quad= \chi\big(X_{t_1}^{t_0,x_0} \in [x_1, x_1 + dx_1)\big)\, \chi\big(X_{t_2}^{t_1,x_1} \in [x_2, x_2 + dx_2)\big) \times \cdots \\ &\qquad\qquad \cdots \times \chi\big(X_{t_n}^{t_{n-1},x_{n-1}} \in [x_n, x_n + dx_n)\big) \end{aligned} \tag{15.26}$$

Therefore (15.24) becomes

$$\begin{aligned}
&\mathsf{P}_W\Big(X_{t_1} \in [x_1, x_1+dx_1), ..., X_{t_n} \in [x_n, x_n+dx_n)\Big) \qquad (15.27)\\
&= \int \chi\big(X^{t_0,x_0}_{t_1} \in [x_1, x_1+dx_1)\big)\, \chi\big(X^{t_1,x_1}_{t_2} \in [x_2, x_2+dx_2)\big) \times \cdots \\
&\qquad \cdots \times \chi\big(X^{t_{n-1},x_{n-1}}_{t_n} \in [x_n, x_n+dx_n)\big) \prod_{j=1}^n dW^{y_{t_{j-1}}}_{(t_{j-1},t_j]}\big(\{y_t\}_{t_{j-1}<t\le t_j}\big) \\
&\overset{(15.23)}{=} \int \prod_{j=1}^n \Big\{\chi\big(X^{t_{j-1},x_{j-1}}_{t_j} \in [x_j, x_j+dx_j)\big)\, dW^{y_{t_{j-1}}}_{(t_{j-1},t_j]}\big(\{y_t\}_{t_{j-1}<t\le t_j}\big)\Big\} \\
&= \prod_{j=1}^n \mathsf{P}_W\big(X^{t_{j-1},x_{j-1}}_{t_j} \in [x_j, x_j+dx_j)\big) \\
&= p(t_0,x_0;t_1,x_1)\, p(t_1,x_1;t_2,x_2)\cdots p(t_{n-1},x_{n-1};t_n,x_n)\, dx_1 dx_2\cdots dx_n
\end{aligned}$$

Here we used the fact that, for example,

$$\int \chi\Big(X^{t_{n-1},x_{n-1}}_{t_n} \in [x_n, x_n+dx_n)\Big)\, dW^{y_{t_{n-1}}}_{(t_{n-1},t_n]}\big(\{y_t\}_{t_{n-1}<t\le t_n}\big) \qquad (15.28)$$

does not depend on $y_{t_{n-1}}$. This proves the lemma ■

The Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process is the Ito diffusion which is the solution of the SDE

$$dX_t = -c\, X_t\, dt + \sigma\, dB_t \qquad (15.29)$$

where c and σ are some constants. It can be solved explicitely. To this end consider $Y_t = e^{ct} X_t$ whose differential is given by

$$\begin{aligned}
dY_t &= c\, e^{ct} X_t\, dt + e^{ct} dX_t + 0 \\
&= \sigma\, e^{ct} dB_t \qquad (15.30)
\end{aligned}$$

Thus $Y_t = Y_0 + \sigma \int_0^t e^{cs} dB_s$ or $(X_0 = x)$

$$X_t = x\, e^{-ct} + \sigma \int_0^t e^{-c(t-s)} dB_s \qquad (15.31)$$

In particular,

$$\begin{aligned}
X_{t+h} &= e^{-ch} X_t + \sigma \int_t^{t+h} e^{-c(t+h-s)} dB_s \\
&= e^{-ch} X_t + \sigma \int_0^h e^{-c(h-s)} dB_{s+t} \qquad (15.32)
\end{aligned}$$

which shows explicitely the Markov property of X_t. By Lemma 12.3, the Ornstein-Uhlenbeck process is a Gaussian process with mean

$$\mathsf{E}_W[X_t] \;=\; x\,e^{-ct} \tag{15.33}$$

and variance

$$\mathsf{V}_W[X_t] \;=\; \sigma^2 \int_0^t e^{-2c(t-s)} ds \;=\; \tfrac{\sigma^2}{2c}(1-e^{-2ct}) \tag{15.34}$$

Thus,

$$\mathsf{P}_W\big(X_t \le a\big) \;=\; \int_{-\infty}^{a} \frac{1}{\sqrt{\pi \frac{\sigma^2}{c}(1-e^{-2ct})}}\; e^{-\frac{(y-xe^{-ct})^2}{\frac{\sigma^2}{c}(1-e^{-2ct})}}\, dy \tag{15.35}$$

In particular, since X_t is time homogenous as the coefficients in (15.29) do not explicitely depend on t,

$$\begin{aligned} p(s,x;t,y)\,dy \;&=\; \mathsf{P}_W\big(X_t^{s,x} \in [y, y+dy)\big) \\ &=\; \mathsf{P}_W\big(X_{t-s}^{0,x} \in [y, y+dy)\big) \end{aligned} \tag{15.36}$$

which gives the Mehler kernel

$$p(s,x;t,y) \;=\; \frac{1}{\sqrt{\pi \frac{\sigma^2}{c}(1-e^{-2c(t-s)})}}\; e^{-\frac{(y-xe^{-c(t-s)})^2}{\frac{\sigma^2}{c}(1-e^{-2c(t-s)})}} \tag{15.37}$$

Further Examples

Consider the SDE (see [12], excercise 5.16)

$$dX_t \;=\; f(t,X_t)\,dt + c(t)X_t dB_t \tag{15.38}$$

Define the 'integrating factor'

$$F_t \;=\; e^{-\int_0^t c(s)dB_s + \frac12 \int_0^t c^2(s)ds} \tag{15.39}$$

and let

$$Y_t \;:=\; F_t\,X_t \tag{15.40}$$

Because of ($h_t := \int_0^t c(s)dB_s$)

$$\begin{aligned} dF_t \;&=\; \tfrac{\partial F}{\partial t}\,dt + \tfrac{\partial F}{\partial h}\,dh_t + \tfrac12 \tfrac{\partial^2 F}{\partial h^2}\,dh_t^2 \\ &=\; \tfrac{c_t^2}{2}\,F_t\,dt - F_t\,c_t\,dB_t + \tfrac12 F_t\,c_t^2\,dt \\ &=\; c_t^2\,F_t\,dt - F_t\,c_t\,dB_t \end{aligned} \tag{15.41}$$

we have (the first two lines of (15.42) being more mnemonic)

$$\begin{aligned} d\langle F, X\rangle_t &= dF_t \cdot dX_t \\ &= -F_t\, c_t\, dB_t \cdot c_t\, X_t\, dB_t \\ &= -c_t^2\, F_t\, X_t\, dt \end{aligned} \tag{15.42}$$

Thus,

$$\begin{aligned} dY_t &= dF_t\, X_t + F_t\, dX_t + d\langle F, X\rangle_t \\ &= c_t^2\, F_t\, X_t\, dt - c_t\, F_t\, X_t\, dB_t + f(t, X_t)\, F_t\, dt + c_t\, F_t\, X_t dB_t - c_t^2\, F_t\, X_t\, dt \\ &= f(t, X_t)\, F_t\, dt \\ &= f(t, F_t^{-1} Y_t)\, F_t\, dt \end{aligned} \tag{15.43}$$

In particular,

$$d\langle Y\rangle_t = 0 \tag{15.44}$$

As a concrete example consider the SDE

$$dX_t = \mu\, X_t^\gamma\, dt + \sigma\, X_t dB_t \tag{15.45}$$

For $\gamma = 1$ the solution is a geometric Brownian, so let $\gamma \neq 1$. Then

$$F_t = e^{-\sigma B_t + \frac{\sigma^2}{2} t} \tag{15.46}$$

and (15.43) becomes

$$\begin{aligned} dY_t &= \mu F_t^{-\gamma} Y_t^\gamma\; F_t dt \\ \tfrac{dY_t}{Y_t^\gamma} &= \mu\, e^{(1-\gamma)(-\sigma B_t + \frac{\sigma^2}{2} t)}\, dt \\ \tfrac{1}{1-\gamma}(Y_t^{1-\gamma} - Y_0^{1-\gamma}) &= \mu \int_0^t e^{(1-\gamma)(-\sigma B_s + \frac{\sigma^2}{2} s)} ds \end{aligned} \tag{15.47}$$

which gives

$$Y_t = \left\{ Y_0^{1-\gamma} + (1-\gamma)\mu \int_0^t e^{(1-\gamma)(-\sigma B_s + \frac{\sigma^2}{2} s)} ds \right\}^{\frac{1}{1-\gamma}}$$

or

$$X_t = X_0\, e^{\sigma B_t - \frac{\sigma^2}{2} t} \left\{ 1 + (1-\gamma)\mu X_0^{\gamma-1} \int_0^t e^{(1-\gamma)\left(-\sigma B_s + \frac{\sigma^2}{2} s\right)} ds \right\}^{\frac{1}{1-\gamma}} \tag{15.48}$$

For $\gamma \to 1$ this converges to

$$\begin{aligned} X_t &\to X_0\, e^{\sigma B_t - \frac{\sigma^2}{2} t} \left\{ 1 + (1-\gamma)\mu \int_0^t ds \right\}^{\frac{1}{1-\gamma}} \\ &\to X_0\, e^{\sigma B_t + (\mu - \frac{\sigma^2}{2}) t} \end{aligned}$$

which is geometric Brownian motion as it should be. The Markov property of X_t can be seen from

$$X_{t+h}^{1-\gamma} \;=\; e^{(1-\gamma)\left(\sigma(B_{t+h}-B_t)-\frac{\sigma^2}{2}h\right)}\,X_t^{1-\gamma} \;+\; (1-\gamma)\mu\int_t^{t+h} e^{(1-\gamma)(-\sigma B_s+\frac{\sigma^2}{2}s)}ds \qquad (15.49)$$

which shows that, once X_t is fixed, X_{t+h} does not depend on $\{X_s\}_{0\leq s<t}$ or $\{B_s\}_{0\leq s<t}$.