Chapter 23

The Heston Model

In the last chapter we considered general stochastic volatility models whose price process is given by the solution of the SDE system

$$dS_t = \mu_t \, S_t \, dt + \sqrt{\nu_t} \, S_t \, dB_t^1 \tag{23.1}$$

$$d\nu_t = \alpha(S_t, \nu_t, t) \, dt + \beta(S_t, \nu_t, t) \, \sqrt{\nu_t} \, dB_t^2 \tag{23.2}$$

where B_t^1 and B_t^2 are two Brownian motions with correlation $\rho \in (-1, 1)$. The Heston model is given by the choice

$$\alpha(S_t, \nu_t, t) = \kappa(\bar{\nu} - \nu_t) \tag{23.3}$$

$$\beta(S_t, \nu_t, t) = \beta \tag{23.4}$$

where κ, $\bar{\nu}$ and β are constants. That is, the volatility is given by a Cox-Ingersoll-Ross process. The Heston model has become popular because it is explicitely solvable, its generating or characteristic function can be computed explicitely. As a consequence, also the pricing PDE for european options with payoffs $H(S_T)$ can be solved explicitely [8].

Recall the general pricing formula of the last chapter. If H is some (probably exotic) european option with payoff $H\big(\{S_t\}_{t_0 \le t \le T}\big)$, then the price at time t_0 is given by

$$V_{t_0} = e^{-r(T-t_0)} \int H\big(\{S_t\}_{t_0 \le t \le T}\big) \, dW_{(t_0,T]}(y^1, y^2) \tag{23.5}$$

where

$$S_t = S_{t_0} \, e^{\int_{t_0}^t \sqrt{\nu_s} dy_s^1 + \int_{t_0}^t (r - \frac{\nu_s}{2}) ds} \tag{23.6}$$

and ν is a solution of the SDE

$$d\nu = -\tilde{\phi} \, dt + \beta \sqrt{\nu} \big(\rho \, dy^1 + \sqrt{1-\rho^2} \, dy^2\big) \tag{23.7}$$

where $\tilde{\phi}$ is the universal function given by (22.51). Introduce the variable

$$x_t \;:=\; \log\left[e^{-r(t-t_0)}\frac{S_t}{S_{t_0}}\right] \;=\; \int_{t_0}^t \sqrt{\nu_s}\,dy_s^1 - \int_{t_0}^t \frac{\nu_s}{2}\,ds \tag{23.8}$$

such that

$$S_t = S_{t_0}\,e^{x_t + r(t-t_0)} \tag{23.9}$$

and let

$$h\big(\{x_t\}_{t_0\le t\le T}\big) \;:=\; e^{-r(T-t_0)} H\Big(\big\{S_{t_0}\,e^{x_t+r(t-t_0)}\big\}_{t_0\le t\le T}\Big) \tag{23.10}$$

Then we can write

$$V_{t_0} \;=\; \int h\big(\{x_t\}_{t_0\le t\le T}\big)\,dW_{(t_0,T]}(y^1,y^2) \tag{23.11}$$

The integral (23.11) can be computed, at least in principle, if we know the finite dimensional distributions ($x_{t_0}=0$)

$$\mathsf{P}\Big(x_{t_1}\in[x_1,x_1+dx_1),\ldots,x_{t_n}\in[x_n,x_n+dx_n)\Big) \;=:\; p(t_0,0;t_1,x_1;\ldots;t_n,x_n)\,dx_1\cdots dx_n \tag{23.12}$$

These in turn can be computed from the generating functional

$$G\big(\{\lambda_t\}_{t_0\le t\le T}\big) \;:=\; \mathsf{E}\Big[e^{i\int_{t_0}^T \lambda_t\,dx_t}\Big] \tag{23.13}$$

where the pair (x_t,ν_t) is a solution of the SDE system

$$dx_t \;=\; -\frac{\nu_t}{2}\,dt + \sqrt{\nu_t}\,dB_t^1 \tag{23.14}$$

$$d\nu_t \;=\; -\tilde{\phi}_t\,dt + \beta_t\sqrt{\nu_t}\,dB_t^2\,, \qquad dB_t^1\cdot dB_t^2 = \rho\,dt \tag{23.15}$$

For example, if we choose $\lambda_s = \lambda\,\chi(t_0\le s\le t)$, then

$$G\big(\{\lambda_t\}\big) \;\equiv\; G_t(\lambda) \;=\; \mathsf{E}\big[\,e^{i\lambda(x_t-x_{t_0})}\,\big] \;=\; \mathsf{E}\big[\,e^{i\lambda x_t}\,\big] \;=\; \int_{\mathbb{R}} e^{i\lambda y}\,p(t_0,0;t,y)\,dy \tag{23.16}$$

such that

$$p(t_0,0;t,y) \;=\; \int_{\mathbb{R}} e^{-i\lambda y}\,G_t(\lambda)\,\frac{d\lambda}{2\pi} \tag{23.17}$$

is obtained as the Fourier transform of the generating function $G_t(\lambda)$. Similarly the higher dimensional distributions (23.12) are obtained as higher dimensional Fourier transforms. For example the choice

$$\lambda_s := (\lambda_1+\lambda_2)\,\chi(t_0\le s<t_1) + \lambda_2\,\chi(t_1\le s<t_2) \tag{23.18}$$

leads to

$$G(\{\lambda_t\}) \equiv G_{t_1,t_2}(\lambda_1,\lambda_2)) = \mathsf{E}\big[e^{i(\lambda_1+\lambda_2)(x_{t_1}-x_{t_0})+i\lambda_2(x_{t_2}-x_{t_1})} \big] = \mathsf{E}\big[e^{i(\lambda_1 x_{t_1}+\lambda_2 x_{t_2})} \big]$$
$$= \int_{\mathbb{R}^2} e^{i(\lambda_1 y_1+\lambda_2 y_2)}\, p(t_0,0;t_1,y_1;t_2y_2)\, dy_1 dy_2 \tag{23.19}$$

such that

$$p(t_0,0;t_1,y_1;t_2,y_2) = \int_{\mathbb{R}^2} e^{-i(\lambda_1 y_1+\lambda_2 y_2)}\, G_{t_1,t_2}(\lambda_1,\lambda_2)\, \tfrac{d\lambda_1}{2\pi}\tfrac{d\lambda_2}{2\pi} \tag{23.20}$$

Let us now turn to the evaluation of (23.13). There is the following

Theorem 23.1: Let (x_t,ν_t) be a solution of the SDE system

$$dx_t = -\tfrac{\nu_t}{2}\,dt + \sqrt{\nu_t}\,dB_t^1 \tag{23.21}$$
$$d\nu_t = \psi_t\,dt + \beta\sqrt{\nu_t}\,dB_t^2\,, \qquad dB_t^1\cdot dB_t^2 = \rho\,dt \tag{23.22}$$

with initial conditions $x_{t_0}=0$, $\nu_{t_0}=\nu_0$. Here β is a positive constant and ψ_t is some function which does not depend on the Brownian motion B_t^1. Consider the generating functional

$$G(\{\lambda_t\}_{t_0\le t\le T}) := \mathsf{E}\Big[e^{i\int_{t_0}^T \lambda_t\,dx_t} \Big] \tag{23.23}$$

Then

a) For general ψ_t, λ_t, the function G is given by

$$G(\{\lambda_t\}_{t_0\le t\le T}) = e^{-i\frac{\rho}{\beta}\lambda_{t_0}\nu_{t_0}} \int e^{i\frac{\rho}{\beta}\lambda_T\nu_T-\int_{t_0}^T\left[\frac{1}{2}(i\lambda_s+(1-\rho^2)\lambda_s^2)+i\frac{\rho}{\beta}\lambda_s'\right]\nu_s\,ds}\, e^{-i\frac{\rho}{\beta}\int_{t_0}^T\lambda_s\psi_s ds} dW(y) \tag{23.24}$$

where $dW(y)$ is the one dimensional Wiener measure and ν_t is a solution of

$$d\nu_t = \psi_t\,dt + \beta\sqrt{\nu_t}\,dy_t \tag{23.25}$$

b) For constant $\lambda_t\equiv\lambda$ and

$$\psi_t = \kappa(\bar\nu-\nu_t) \tag{23.26}$$

with initial condition $\nu_0=\nu_{t_0}$, (23.24) becomes

$$G(\lambda) = e^{-i\frac{\rho}{\beta}\lambda[\nu_{t_0}+\kappa\bar\nu(T-t_0)]}\, e^{\frac{\kappa+2f'/f(t_0)}{\beta^2}\nu_{t_0}+\frac{\kappa^2}{\beta^2}\bar\nu(T-t_0)}\, e^{\frac{2\kappa\bar\nu}{\beta^2}\left\{\log[\frac{r_f(t)}{r_f(0)}]+i\left(\varphi_f(t)-\varphi_f(0)\right)\right\}} \tag{23.27}$$

where the function f is given by

$$f(s) = \sqrt{\xi}\cosh[\sqrt{\xi}(T-s)] - \gamma\sinh[\sqrt{\xi}(T-s)] \tag{23.28}$$
$$=: r_f(s)\, e^{i\varphi_f(s)}$$

with a differentiable φ_f and constants

$$\gamma = -\tfrac{\kappa}{2} + i\tfrac{\rho\beta\lambda}{2} \tag{23.29}$$
$$\xi = \tfrac{(1-\rho^2)\lambda^2\beta^2+\kappa^2}{4} + i\lambda\tfrac{\beta^2-2\beta\rho\kappa}{4}\,. \tag{23.30}$$

and the square root can be choosen to be any complex square root since G depends only on $(\sqrt{\xi})^2 = \xi$.

Proof: We rewrite the SDE system (23.14,23.15) in terms of two uncorrelated Brownian motions y_t^1 and y_t^2,

$$dx_t = -\tfrac{\nu_t}{2}\,dt + \sqrt{\nu_t}\left(\sqrt{1-\rho^2}\,dy_t^1 + \rho\,dy_t^2\right) \tag{23.31}$$
$$d\nu_t = \psi_t\,dt + \beta_t\sqrt{\nu_t}\,dy_t^2 \tag{23.32}$$

We have to compute the integral

$$\int e^{i\int_{t_0}^T \lambda_s\,dx_s} dW(y^1)\,dW(y^2)$$
$$= \int e^{-i\int_{t_0}^T \lambda_s \frac{\nu_s}{2}\,ds + i\int_{t_0}^T \lambda_s\left(\sqrt{1-\rho^2}\sqrt{\nu_s}\,dy_s^1 + \rho\sqrt{\nu_s}\,dy_s^2\right)} dW(y^1)\,dW(y^2)$$
$$= \int e^{-i\int_{t_0}^T \lambda_s \frac{\nu_s}{2}\,ds + i\int_{t_0}^T \lambda_s\rho\sqrt{\nu_s}\,dy_s^2}\left\{\int e^{i\int_{t_0}^T \lambda_s\sqrt{1-\rho^2}\sqrt{\nu_s}\,dy_s^1} dW(y^1)\right\} dW(y^2) \tag{23.33}$$

Since by assumption ψ does not depend on y^1, equation (23.32) determines ν as a function of y^2 only, ν does not depend on y^1. Thus we can perform the y^1-integral in the wavy brackets of (23.33). Using Lemma..., we obtain

$$\int e^{i\int_{t_0}^T \lambda_s\sqrt{1-\rho^2}\sqrt{\nu_s}\,dy_s^1} dW(y^1) = \int_{\mathbb{R}} e^{iy}\frac{1}{\sqrt{2\pi(T-t_0)\bar{\sigma}^2}}\,e^{-\frac{y^2}{2(T-t_0)\bar{\sigma}^2}}\,dy \tag{23.34}$$

where

$$(T-t_0)\bar{\sigma}^2 = \int_{t_0}^T \lambda_s^2(1-\rho^2)\nu_s\,ds \tag{23.35}$$

Thus,

$$\int e^{i\int_{t_0}^T \lambda_s\sqrt{1-\rho^2}\sqrt{\nu_s}\,dy_s^1} dW(y^1) = e^{-\frac{(T-t_0)\bar{\sigma}^2}{2}}$$
$$= e^{-\frac{1-\rho^2}{2}\int_{t_0}^T \lambda_s^2\nu_s\,ds} \tag{23.36}$$

Substituting (23.36) into (23.33),we arrive at

$$\begin{aligned}
\mathsf{E}\big[e^{i\int_{t_0}^T \lambda_s\,dx_s}\big] &= \int e^{-i\int_{t_0}^T \lambda_s \frac{\nu_s}{2}\,ds + i\int_{t_0}^T \lambda_s\rho\sqrt{\nu_s}\,dy_s^2} e^{-\frac{1-\rho^2}{2}\int_{t_0}^T \lambda_s^2 \nu_s\,ds}\,dW(y^2) \\
&= \int e^{-\frac{1}{2}\int_{t_0}^T \left[i\lambda_s + (1-\rho^2)\lambda_s^2\right]\nu_s\,ds}\, e^{i\int_{t_0}^T \lambda_s\rho\sqrt{\nu_s}\,dy_s}\,dW(y) \qquad (23.37)
\end{aligned}$$

where we renamed $y_t^2 \to y_t$ in the last line. Since ν_t is the solution of the SDE

$$d\nu_t = \psi_t\,dt + \beta\sqrt{\nu_t}\,dy_t \qquad (23.38)$$

we have

$$\begin{aligned}
i\int_{t_0}^T \lambda_s\rho\sqrt{\nu_s}\,dy_s &= i\tfrac{\rho}{\beta}\int_{t_0}^T \lambda_s[d\nu_s - \psi_s\,ds] \\
&= i\tfrac{\rho}{\beta}\big(\lambda_T\nu_T - \lambda_{t_0}\nu_{t_0}\big) - i\tfrac{\rho}{\beta}\int_{t_0}^T [\nu_s d\lambda_s + \lambda_s\psi_s\,ds] \\
&= i\tfrac{\rho}{\beta}\big(\lambda_T\nu_T - \lambda_{t_0}\nu_{t_0}\big) - i\tfrac{\rho}{\beta}\int_{t_0}^T [\nu_s\lambda_s' + \lambda_s\psi_s\,]\,ds \qquad (23.39)
\end{aligned}$$

Substituting this in (23.37) gives

$$\mathsf{E}\big[e^{i\int_{t_0}^T \lambda_s\,dx_s}\big] = e^{-i\frac{\rho}{\beta}\lambda_{t_0}\nu_{t_0}} \int e^{i\frac{\rho}{\beta}\lambda_T\nu_T - \int_{t_0}^T\left[\frac{1}{2}(i\lambda_s + (1-\rho^2)\lambda_s^2) + i\frac{\rho}{\beta}\lambda_s'\right]\nu_s\,ds} e^{-i\frac{\rho}{\beta}\int_{t_0}^T \lambda_s\psi_s\,ds} dW(y) \qquad (23.40)$$

This proves part (a). For $\psi_t = \kappa(\bar\nu - \nu_t)$ and constant $\lambda_s \equiv \lambda$, this reads

$$\begin{aligned}
G(\lambda) &= \mathsf{E}\big[e^{i\lambda\int_{t_0}^T dx_s}\big] = \mathsf{E}\big[e^{i\lambda x_T}\big] \\
&= e^{-i\frac{\rho}{\beta}\lambda[\nu_0 + \kappa\bar\nu(T-t_0)]} \int e^{\xi\nu_T(y) - \mu\int_{t_0}^T \nu_s(y)\,ds} dW(y) \qquad (23.41)
\end{aligned}$$

where

$$\begin{aligned}
\xi &= i\tfrac{\rho}{\beta}\lambda \qquad (23.42) \\
\mu &= \tfrac{1-\rho^2}{2}\lambda^2 + i\tfrac{\lambda}{2}\left(1 - 2\tfrac{\rho}{\beta}\kappa\right) \qquad (23.43)
\end{aligned}$$

and now ν is the square root process given by $d\nu_t = \kappa(\bar\nu - \nu_t)dt + \beta\sqrt{\nu_t}\,dy_t$. This expectation has been computed in Corollary 21.3 where we found

$$G(\lambda) = e^{-i\frac{\rho}{\beta}\lambda[\nu_{t_0} + \kappa\bar\nu(T-t_0)]}\, e^{\frac{\kappa + 2f'/f(t_0)}{\beta^2}\nu_0 + \frac{\kappa^2}{\beta^2}\bar\nu t}\, e^{\frac{2\kappa\bar\nu}{\beta^2}\left\{\log[\frac{r_f(t)}{r_f(0)}] + i\left(\varphi_f(t) - \varphi_f(0)\right)\right\}} \qquad (23.44)$$

where the function f is given by

$$f(s) = \sqrt{\xi}\cosh[\sqrt{\xi}(T-s)] - \gamma\sinh[\sqrt{\xi}(T-s)]$$

with

$$
\begin{aligned}
\gamma &= -\tfrac{\kappa}{2} + i\tfrac{\rho\beta\lambda}{2} && (23.45)\\
\xi &= \tfrac{(1-\rho^2)\lambda^2\beta^2+\kappa^2}{4} + i\lambda\tfrac{\beta^2-2\beta\rho\kappa}{4}\,. && (23.46)
\end{aligned}
$$

This proves the theorem. ■

Now, having computed the generating function, we can compute the density $p(t_0, 0; t, y)$ according to (23.17). Then the price of some plain vanilla european option is given by

$$
\begin{aligned}
V_{t_0} &= \int h\big(x_T\big)\, dW_{(t_0,T]}(y^1, y^2)\\
&= \int_{\mathbb{R}} h(y)\, p(t_0, 0; T, y)\, dy\\
&= \int_{\mathbb{R}} h(y) \int_{\mathbb{R}} e^{-i\lambda y}\, G(\lambda)\, \tfrac{d\lambda}{2\pi}\, dy && (23.47)
\end{aligned}
$$

If the payoff $h(y)$ would have a finite Fourier transform $\hat{h}(\lambda)$, we could interchange the integrals to obtain

$$
V_{t_0} = \int_{\mathbb{R}} \hat{h}(\lambda)\, G(\lambda)\, \tfrac{d\lambda}{2\pi} \qquad (23.48)
$$

However, for a european call $H(S_T) = \max\{S_T - K, 0\}$ we have

$$
h(x) = e^{-r(T-t_0)} H(S_{t_0} e^{x+r(T-t_0)}) = e^{-r(T-t_0)} \max\{S_{t_0} e^{x+r(T-t_0)} - K, 0\} \qquad (23.49)
$$

which apparently does not decay for $x \to \infty$, thus $\hat{h}(\lambda)$ does not exist.

This problem can be circumvented in the following way [2]. By (23.16), the generating function $G(\lambda)$ is the Fourier transform of the density $p(y)$. If $p(y)$ decays like e^{-cy^2} or at least like

$$
p(y) \sim e^{-c|y|^{1+\delta}} \qquad \text{as } \; y \to \pm\infty \qquad (23.50)
$$

for positive constants c and δ, then the generating function $G(\lambda)$ is not only defined on the real axis, but on the whole complex plane since

$$
G_t(\lambda \pm i\alpha) = \int_{\mathbb{R}} e^{-i\lambda y \pm \alpha y} p(y)\, dy \qquad (23.51)
$$

is finite if (23.50) holds. Thus we can write

$$
\begin{aligned}
V_{t_0} &= \int_{\mathbb{R}} h(y)\, p(y)\, dy\\
&= \int_{\mathbb{R}} e^{-\alpha y} h(y)\, e^{\alpha y} p(y)\, dy\\
&= \int_{\mathbb{R}} \big[e^{-\alpha\cdot} h(\cdot)\big]^{\widehat{}}(\lambda)\, \big[e^{\alpha\cdot} p(\cdot)\big]^{\widehat{}}(\lambda)\, \tfrac{d\lambda}{2\pi}\\
&= \int_{\mathbb{R}} \big[e^{-\alpha\cdot} h(\cdot)\big]^{\widehat{}}(\lambda)\, G(\lambda + i\alpha)\, \tfrac{d\lambda}{2\pi} && (23.52)
\end{aligned}
$$

where we used the unitarity of the Fourier transform in the third line of (23.52). For the payoff (23.49), with $\tau = T - t_0$,

$$\begin{aligned} h(x) &= e^{-r\tau}\max\{S_{t_0}e^{x+r\tau} - K, 0\} \;=\; S_{t_0}\big(e^x - \tfrac{K}{S_{t_0}}e^{-r\tau}\big)_+ \\ &= S_{t_0}\big(e^x - e^k\big)_+\,, \qquad k = \log\big[\tfrac{K}{S_{t_0}}e^{-r\tau}\big] \end{aligned} \tag{23.53}$$

one obtains

$$\begin{aligned} \big[e^{-\alpha\cdot}h(\cdot)\big]\widehat{}(\lambda) &= \int_{\mathbb{R}} e^{-i\lambda x}e^{-\alpha x}h(x)\,dx \\ &= S_{t_0}\int_k^\infty e^{-(i\lambda+\alpha)x}(e^x - e^k)\,dx \\ &= S_{t_0}\Big\{\tfrac{1}{-\alpha+1-i\lambda}e^{-(i\lambda+\alpha-1)x}\Big|_k^\infty - \tfrac{1}{-\alpha-i\lambda}e^k e^{-(i\lambda+\alpha)x}\Big|_k^\infty\Big\} \\ &\overset{\alpha\geq 1}{=} S_{t_0}\Big\{-\tfrac{1}{-\alpha+1-i\lambda}e^{-(i\lambda+\alpha-1)k} + \tfrac{1}{-\alpha-i\lambda}e^{-(i\lambda+\alpha-1)k}\Big\} \\ &= S_{t_0}\,\tfrac{1}{(\alpha-1+i\lambda)(\alpha+i\lambda)}\,e^{-(i\lambda+\alpha-1)k} \end{aligned} \tag{23.54}$$

such that the option price is given by

$$V_t \;=\; S_t e^{-(\alpha-1)k}\int_{\mathbb{R}} \tfrac{1}{(\alpha-1+i\lambda)(\alpha+i\lambda)}\,e^{-i\lambda k}\,G(\lambda+i\alpha)\,\tfrac{d\lambda}{2\pi} \tag{23.55}$$

where α has to be choosen bigger than 1. We summarize our results in the follwoing

Theorem 23.2: Let $G(\lambda)$ be the generating function for the Heston model given by (23.27) and let

$$k = \log\big[\tfrac{K}{S_t}e^{-r(T-t)}\big] \tag{23.56}$$

Then the price at time $t \leq T$ of the european call with payoff $\max\{S_T - K, 0\}$ is given by

$$V_t \;=\; S_t e^{-(\alpha-1)k}\int_{\mathbb{R}} \tfrac{1}{(\alpha-1+i\lambda)(\alpha+i\lambda)}\,e^{-i\lambda k}\,G(\lambda+i\alpha)\,\tfrac{d\lambda}{2\pi} \tag{23.57}$$

where α can be choosen to be any real number bigger than 1 such that $G(\lambda + i\alpha)$ exists.