Chapter 17

The Feynman-Kac Formula

The Feynman-Kac formula states that a probabilistic expectation value with respect to some Ito-diffusion can be obtained as a solution of an associated PDE. It may be formulated as follows:

Let $X_t = (X_t^1, ..., X_t^d)$ be a stochastic process which is a solution of the system of stochastic differential equations

$$dX_t^i \;=\; \mu_i(t, X_t)\, dt \;+\; \sigma_i(t, X_t)\, dB_t^i \tag{17.1}$$

where $B_t^1, ..., B_t^d$ are Brownian motions with correlation

$$dB_t^i dB_t^j = \rho_{ij}\, dt \tag{17.2}$$

Let $H(x) = H(x_1, ..., x_d)$ be some payoff. Then the function

$$u(t, x) \;:=\; \mathsf{E}\big[H(X_T)\,|\,X_t = x\big] \tag{17.3}$$

is a solution of the PDE

$$\frac{\partial u}{\partial t} \;+\; \sum_{i=1}^{d} \mu(t, x)\, \frac{\partial u}{\partial x_i} \;+\; \frac{1}{2} \sum_{i,j=1}^{d} \rho_{ij}\sigma_i(t, x)\sigma_j(t, x)\, \frac{\partial^2 u}{\partial x_i \partial x_j} \;=\; 0 \tag{17.4}$$

with final condition

$$u(T, x) = H(x) \tag{17.5}$$

We prove a slightly more general result which allows also the interest rates to depend on X_t. We follow the exposition of [10]. The proof is based on the following

Proposition 17.1: Let $X_t = (X_t^1, ..., X_t^d)$ be an Ito-diffusion given by the SDE

$$dX_t \;=\; \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \tag{17.6}$$

where μ takes values in $\mathbb{R}^d$, σ takes values in $\mathbb{R}^{d\times m}$ and B_t is an m-dimensional (uncorrelated) Brownian motion. Let A be the generator of (17.6), given by ($f = f(t,x) : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$)

$$(Af)(t,x) \;=\; \textstyle\sum_{i=1}^d \mu_i(t,x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^d (\sigma\sigma^T)_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{17.7}$$

Let $u \in C^{1,2}\big(\mathbb{R}^+ \times \mathbb{R}^d\big)$ and $r \in C^0\big(\mathbb{R}^+ \times \mathbb{R}^d\big)$, then the process

$$M_t \;:=\; e^{-\int_0^t r(s,X_s)ds}u(t,X_t) \;-\; \int_0^t e^{-\int_0^s r(v,X_v)dv}\left(\tfrac{\partial u}{\partial t} + Au - ru\right)(s,X_s)\,ds \tag{17.8}$$

is a martingale.

Proof: The statement follows by applying the Ito lemma to the function

$$F_t \;:=\; e^{-\int_0^t r(s,X_s)ds}u(t,X_t) \tag{17.9}$$

Namely, since the covariation $\big\langle e^{-\int_0^s r(v,X_v)dv}, u(s,X_s)\big\rangle = 0$ because $e^{-\int_0^s r(v,X_v)dv}$ is differentiable and therefore of bounded variation,

$$\begin{aligned}
F_t \;&=\; F_0 + \int_0^t \Big(d\big(e^{-\int_0^s r(v,X_v)dv}\big)u(s,X_s) + e^{-\int_0^s r(v,X_v)dv}du(s,X_s) + 0\Big) \\
&=\; u(0,X_0) + \int_0^t \Big(e^{-\int_0^s r(v,X_v)dv}u(s,X_s)\left(-r(s,X_s)\right)ds + e^{-\int_0^s r(v,X_v)dv}\tfrac{\partial u}{\partial t}\,ds\Big) \;+ \\
&\qquad \int_0^t e^{-\int_0^s r(v,X_v)dv}\Big(\textstyle\sum_{i=1}^d \frac{\partial u}{\partial x_i}dX_s^i + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j}dX_s^i dX_s^j\Big) \\
&=\; u(0,X_0) + \int_0^t e^{-\int_0^s r(v,X_v)dv}\Big\{ -ru + \tfrac{\partial u}{\partial t} + \textstyle\sum_{i=1}^d \frac{\partial u}{\partial x_i}\,\mu_i(s,X_s)\,ds \;+ \\
&\qquad \textstyle\sum_{i=1}^d \frac{\partial u}{\partial x_i}\sum_k \sigma_{i,k}(s,X_s)\,dB_s^k + \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2 u}{\partial x_i \partial x_j}\sum_{k,l}\sigma_{i,k}\sigma_{j,l}\,\underbrace{dB_s^k dB_s^l}_{=\delta_{k,l}\,ds}\Big\} \\
&=\; u(0,X_0) + \int_0^t e^{-\int_0^s r(v,X_v)dv}\big\{-ru + \tfrac{\partial u}{\partial t} + Au\big\}ds \;+ \int_0^t e^{-\int_0^s r(v,X_v)dv}\,\nabla u\cdot\sigma\cdot dB_s
\end{aligned} \tag{17.10}$$

Since the process $u(0,X_0) + \int_0^t e^{-\int_0^s r(v,X_v)dv}\,\nabla u\cdot\sigma\cdot dB_s$ is a martingale, the proposition follows ■

Theorem 17.2 (Feynman-Kac Formula): Let X_t be an Ito-diffusion given by (17.6) with generator A given by (17.7) with initial condition $X_t^{t,x} = x$. Let $u \in C^{1,2}\big(\mathbb{R}^+ \times \mathbb{R}^d\big)$ satisfy

$$\tfrac{\partial u}{\partial t} + Au - ru \;=\; 0 \tag{17.11}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$ with final condition

$$u(T,x) \;=\; f(x) \tag{17.12}$$

Then

$$\begin{aligned} u(t,x) &= \mathsf{E}\Big[e^{-\int_t^T r(s,X_s)ds} f(X_T) \,\Big|\, X_t = x\Big] \\ &= \mathsf{E}\Big[e^{-\int_t^T r(s,X_s^{t,x})ds} f(X_T^{t,x})\Big] \end{aligned} \tag{17.13}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

Proof: By proposition 15.1, the process $(M_{t'})_{t\le t'\le T}$ given by

$$\begin{aligned} M_{t'} &= e^{-\int_t^{t'} r(s,X_s^{t,x})ds} u(t',X_{t'}^{t,x}) - \int_t^{t'} e^{-\int_t^s r(v,X_v^{t,x})dv} \left(\tfrac{\partial u}{\partial t} + Au - ru\right)(s,X_s^{t,x})\,ds \\ &\overset{(17.11)}{=} e^{-\int_t^{t'} r(s,X_s^{t,x})ds} u(t',X_{t'}^{t,x}) \end{aligned} \tag{17.14}$$

is a martingale. Thus

$$\begin{aligned} u(t,x) &= M_t \\ &= \mathsf{E}[M_t] \\ &= \mathsf{E}[M_T] \\ &= \mathsf{E}\Big[e^{-\int_t^T r(s,X_s^{t,x})ds} u(T,X_T^{t,x})\Big] \\ &\overset{(17.12)}{=} \mathsf{E}\Big[e^{-\int_t^T r(s,X_s^{t,x})ds} f(X_T^{t,x})\Big] \end{aligned} \tag{17.15}$$

This proves the theorem ■

Remark: By rewriting $M_{t'}$ in (17.14) as

$$\begin{aligned} M_{t'} &= e^{-\int_t^{t'} r(s,X_s^{t,x})ds} u(t',X_{t'}^{t,x}) - \int_t^{t'} e^{-\int_t^s r(v,X_v^{t,x})dv} g(s,X_s^{t,x})\,ds \\ &\quad - \int_t^{t'} e^{-\int_t^s r(v,X_v^{t,x})dv} \left(\tfrac{\partial u}{\partial t} + Au - ru - g\right)(s,X_s^{t,x})\,ds \end{aligned} \tag{17.16}$$

where g is some function, the same reasoning as above shows that

$$v(t,x) \;:=\; \mathsf{E}\Big[e^{-\int_t^T r(s,X_s^{t,x})ds} f(X_T^{t,x}) - \int_t^T e^{-\int_t^s r(v,X_v^{t,x})dv} g(s,X_s^{t,x})\,ds\Big] \tag{17.17}$$

is obtained as a solution of the PDE

$$\tfrac{\partial v}{\partial t} + Av - rv \;=\; g \tag{17.18}$$

with final condition

$$v(T,x) \;=\; f(x) \tag{17.19}$$