Chapter 11

Barrier and Lookback Options in the Black-Scholes Model

We now have all ingredients to derive an analytical formula for the price of a barrier option in the Black-Scholes model. There is the following

Theorem 11.1: Consider the barrier option with payoff

$$H(S_T) = \max\{S_T - K, 0\}\,\chi\big(\min_{0\leq t\leq T} S_t > B\big), \quad K \geq B > 0 \tag{11.1}$$

a 'down-and-out' Barrier-Call with strike K, barrier B and maturity T. Observe that this is only a reasonable payoff if $S_0 > B$. Let the asset price dynamics be given by a Black-Scholes model with drift μ and volatility σ,

$$\tfrac{dS_t}{S_t} \;=\; \mu\,dt \,+\, \sigma\,dx_t$$

or equivalently

$$S_t \;=\; S_t^{(\mu)} \;=\; S_0\,e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t}$$

Let further be $V_{\text{Call}}(S,t)$ be the time-t Black-Scholes price of the call $\max\{S_T - K, 0\}$ given by (6.12) of Theorem 6.1, and let $\kappa := \frac{2r}{\sigma^2}$. Then the time-$t$ price of the barrier option (11.1) is given by

$$V_{\text{Barrier}}(S,t) \;=\; V_{\text{Call}}(S,t) - \big(\tfrac{S}{B}\big)^{1-\kappa}\,V_{\text{Call}}\big(\tfrac{B^2}{S},t\big) \tag{11.2}$$

where $S = S_t$ is the underlying asset price at time t (which is known at time t).

Proof: Recall that $c = \frac{r}{\sigma} - \frac{\sigma}{2}$. From (10.7), (10.10) and (10.23) we get

$$V_{\text{Barrier}}(S_0, t=0) \;=\; e^{-rT}\int H\left(S_0\,e^{\sigma y_T}, S_0\,e^{\sigma\min_{0\leq t\leq T}\{y_t\}}\right) e^{cy_T - \frac{c^2}{2}T} dW(\{y_t\}_{0<t\leq T})$$

$$
\begin{aligned}
&= e^{-rT} \int_{\mathbb{R}^2} \big(S_0\, e^{\sigma a} - K\big)_+ \chi\big(\, S_0\, e^{\sigma b} > B\,\big)\, e^{ca-\frac{c^2}{2}T}\, \rho_{y_T, \min_{0\le t\le T}\{y_t\}}(a,b)\, da\, db \\
&= e^{-rT} \int_{\mathbb{R}^2} \big(S_0\, e^{\sigma a} - K\big)_+ \chi\big(\, S_0\, e^{\sigma b} > B\,\big)\, e^{ca-\frac{c^2}{2}T} \big(-\tfrac{2}{T}\big)\, \varphi'\big(\tfrac{a-2b}{\sqrt{T}}\big)\, \chi(a>b)\, da\, db \\
&= e^{-rT} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a} - K\big)_+ e^{ca-\frac{c^2}{2}T} \int_{\mathbb{R}} db\, \chi\Big(a > b > -\tfrac{1}{\sigma}\log\big(\tfrac{S_0}{B}\big)\Big)\, \big(-\tfrac{2}{T}\big)\, \varphi'\big(\tfrac{a-2b}{\sqrt{T}}\big) \\
&= e^{-rT} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a} - K\big)_+ e^{ca-\frac{c^2}{2}T}\, \chi\Big(a > -\tfrac{1}{\sigma}\log\big(\tfrac{S_0}{B}\big)\Big)\, \tfrac{1}{\sqrt{T}}\, \varphi\big(\tfrac{a-2b}{\sqrt{T}}\big)\Big|_{-\frac{1}{\sigma}\log\left(\frac{S_0}{B}\right)}^{a} \qquad (11.3) \\
&= e^{-rT} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a} - K\big)_+ e^{ca-\frac{c^2}{2}T}\, \tfrac{1}{\sqrt{T}} \Big[\varphi\big(-\tfrac{a}{\sqrt{T}}\big) - \varphi\big(\tfrac{a+\frac{2}{\sigma}\log(\frac{S_0}{B})}{\sqrt{T}}\big)\Big] \\
&= e^{-rT} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a} - K\big)_+ \tfrac{1}{\sqrt{2\pi T}}\, e^{-\frac{(a-cT)^2}{2T}} \\
&\quad - e^{-rT} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a} - K\big)_+ \tfrac{1}{\sqrt{2\pi T}}\, e^{-\frac{\left(a+\frac{2}{\sigma}\log(\frac{S_0}{B})\right)^2}{2T}}\, e^{ca-\frac{c^2}{2}T} \\
&= e^{-rT} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a+\sigma cT} - K\big)_+ \tfrac{1}{\sqrt{2\pi T}}\, e^{-\frac{a^2}{2T}} \\
&\quad - e^{-rT}\big(\tfrac{S_0}{B}\big)^{-\frac{2c}{\sigma}} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a-2\log(\frac{S_0}{B})} - K\big)_+ \tfrac{1}{\sqrt{2\pi T}}\, e^{-\frac{a^2}{2T}}\, e^{ca-\frac{c^2}{2}T} \\
&= e^{-rT} \int_{\mathbb{R}} da\, \big(S_0\, e^{\sigma a+(r-\frac{\sigma^2}{2})T} - K\big)_+ \tfrac{1}{\sqrt{2\pi T}}\, e^{-\frac{a^2}{2T}} \\
&\quad - e^{-rT}\big(\tfrac{S_0}{B}\big)^{-(\kappa-1)} \int_{\mathbb{R}} da\, \big(\tfrac{B^2}{S_0}\, e^{\sigma a} - K\big)_+ \tfrac{1}{\sqrt{2\pi T}}\, e^{-\frac{(a-cT)^2}{2T}} \\
&= V_{\text{Call}}(S_0, 0) - \big(\tfrac{S}{B}\big)^{-(\kappa-1)} V_{\text{Call}}\big(\tfrac{B^2}{S_0}, 0\big) \qquad (11.4)
\end{aligned}
$$

Observe that in (11.3) we used that $K \geq B$ since otherwise a may be less than $\frac{1}{\sigma}\log\frac{B}{S_0}$. This proves the theorem for $t = 0$. For general $t \in [0, T]$, the result follows from the following lemma. ■

Lemma 11.2: Let $dW_{(t,x_t)}^{(t,T]}$ be the Wiener measure for the Brownian motion on the intervall $[t, T]$ starting at x_t at time t. Let

$$S_t^{(\mu)} = S_0\, e^{\sigma x_t + (\mu - \frac{\sigma^2}{2})t} \qquad (11.5)$$

be a geometric Brownian motion and let $d\tilde{W}_{(t,x_t)}^{(t,T]}$ be the equivalent martingale measure defined by the kernels $\tilde{p}$ of (9.14). Let $H\big(\{S_s^{(\mu)}\}_{0\le s\le T}\big) = H\big(\{S_s^{(\mu)}\}_{0\le s\le t}, \{S_s^{(\mu)}\}_{t<s\le T}\big)$ be some payoff. Then

$$
\begin{aligned}
&\int H\Big(\{S_s^{(\mu)}\}_{0\le s\le T}\Big)\, d\tilde{W}_{(t,x_t)}^{(t,T]}(\{x_s\}_{t<s\le T}) \qquad (11.6) \\
&\quad = \int H\Big(\{S_s^{(\mu)}\}_{0\le s\le t}, \{S_t^{(\mu)}\, e^{\sigma y_s + (r-\frac{\sigma^2}{2})s}\}_{0<s\le T-t}\Big)\, dW_{(0,0)}^{(0,T-t]}(\{y_s\}_{0<s\le T-t})
\end{aligned}
$$

Observe that in this formula the asset prices $\{S_s^{(\mu)}\}_{0\leq s\leq t}$ are all known, they have all realized already, since current time is t and not 0.

Proof: As in Theorem 9.2 we have

$$\begin{aligned}
&\int H\Big(\{S_s^{(\mu)}\}_{0\leq s\leq T}\Big)\, d\tilde{W}^{(t,T]}_{(t,x_t)}(\{x_s\}_{t<s\leq T}) \\
&\quad = \int H\Big(\{S_s^{(\mu)}\}_{0\leq s\leq t}, \{S_s^{(\mu)}\}_{t<s\leq T}\Big)\, d\tilde{W}^{(t,T]}_{(t,x_t)}(\{x_s\}_{t<s\leq T}) \\
&\quad = \int H\Big(\{S_s^{(\mu)}\}_{0\leq s\leq t}, \{S_s^{(r)}\}_{t<s\leq T}\Big)\, dW^{(t,T]}_{(t,x_t+\frac{\mu-r}{\sigma}t)}(\{\tilde{x}_s\}_{t<s\leq T}) \qquad (11.7)
\end{aligned}$$

For $s\in(t,T]$ we have with the definition $y_u := \tilde{x}_{u+t} - x_t - \frac{\mu-r}{\sigma}t$, $u>0$,

$$\begin{aligned}
S_s^{(r)} &= S_0\, e^{\sigma\tilde{x}_s + (r-\frac{\sigma^2}{2})s} \\
&= S_0\, e^{\sigma(\tilde{x}_{s-t+t} - x_t - \frac{\mu-r}{\sigma}t) + \sigma x_t + (\mu-r)t + (r-\frac{\sigma^2}{2})s} \\
&= S_0\, e^{\sigma y_{s-t} + \sigma x_t + (\mu-\frac{\sigma^2}{2})t + (r-\frac{\sigma^2}{2})(s-t)} \\
&= S_t^{(\mu)}\, e^{\sigma y_{s-t} + (r-\frac{\sigma^2}{2})(s-t)} \qquad (11.8)
\end{aligned}$$

The Wiener measure in the last line of (11.7) is given by ($s_j = j\Delta t$, $N_t = t/\Delta t$)

$$dW^{(t,T]}_{(t,x_t+\frac{\mu-r}{\sigma}t)}(\{\tilde{x}_s\}_{t<s\leq T}) = \lim_{\Delta t\to 0}\prod_{j=N_t+1}^{N_T}\frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{(\tilde{x}_{s_j}-\tilde{x}_{s_{j-1}})^2}{2\Delta t}}\, d\tilde{x}_{s_j} \qquad (11.9)$$

Observe that on the right hand side of (11.9) the variables $\tilde{x}_{s_{N_t+1}},...,\tilde{x}_{s_{N_T}}$ are simply integration variables whereas $\tilde{x}_{s_{N_t}} = \tilde{x}_t := x_t + \frac{\mu-r}{\sigma}t$ is the starting point of the Brownian motion. Because this starting point is not x_t but $x_t + \frac{\mu-r}{\sigma}t$, we used the tilde for the integration variables. x_t is related to the price process according to (11.5). Making the substitution of variables, for $t<s\leq T$,

$$\begin{aligned}
y_{s-t} &:= \tilde{x}_s - \tilde{x}_t \qquad (11.10) \\
&= \tilde{x}_s - x_t - \tfrac{\mu-r}{\sigma}t \\
dy_{s-t} &= d\tilde{x}_s
\end{aligned}$$

we obtain ($y_0 := 0$)

$$\begin{aligned}
dW^{(t,T]}_{(t,x_t+\frac{\mu-r}{\sigma}t)}(\{\tilde{x}_s\}_{t<s\leq T}) &= \lim_{\Delta t\to 0}\prod_{j=N_t+1}^{N_T}\frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{(y_{s_j-t}-y_{s_{j-1}-t})^2}{2\Delta t}}\, dy_{s_j-t} \\
&= \lim_{\Delta t\to 0}\prod_{j=1}^{N_T-N_t}\frac{1}{\sqrt{2\pi\Delta t}}\, e^{-\frac{(y_{s_j}-y_{s_{j-1}})^2}{2\Delta t}}\, dy_{s_j} \\
&= dW^{(0,T-t]}_{(0,0)}(\{y_s\}_{0<s\leq T-t}) \qquad (11.11)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int H\Big(\{S_s^{(\mu)}\}_{0\leq s\leq T}\Big)\, d\tilde{W}^{(t,T]}_{(t,x_t)}(\{x_s\}_{t<s\leq T}) \\
&\quad= \int H\Big(\{S_s^{(\mu)}\}_{0\leq s\leq t},\{S_s^{(r)}\}_{t<s\leq T}\Big)\, dW^{(t,T]}_{(t,x_t+\frac{\mu-r}{\sigma}t)}(\{\tilde{x}_s\}_{t<s\leq T}) \\
&\quad= \int H\Big(\{S_s^{(\mu)}\}_{0\leq s\leq t},\{S_t^{(\mu)}\, e^{\sigma y_{s-t}+(r-\frac{\sigma^2}{2})(s-t)}\}_{t<s\leq T}\Big)\, dW^{(t,T]}_{(t,x_t+\frac{\mu-r}{\sigma}t)}(\{\tilde{x}_s\}_{t<s\leq T}) \\
&\quad= \int H\Big(\{S_s^{(\mu)}\}_{0\leq s\leq t},\{S_t^{(\mu)}\, e^{\sigma y_s+(r-\frac{\sigma^2}{2})s}\}_{0<s\leq T-t}\Big)\, dW^{(0,T-t]}_{(0,0)}(\{y_s\}_{0<s\leq T-t}) \qquad (11.12)
\end{aligned}$$

which proves the lemma. ■

Lookback Options

We compute the price of a lookback put with payoff

$$H\big(\{S_s\}_{0\leq s\leq T}\big) \;=\; \max_{0\leq s\leq T}\{S_s\} \;-\; S_T \qquad (11.13)$$

where $S_s = S_s^{(\mu)} = S_0\, e^{\sigma x_s+(\mu-\frac{\sigma^2}{2})s}$ is a geometric Brownian motion. The price at time $t\in(0,T)$ is given by

$$V_t \;=\; e^{-r(T-t)}\,\mathsf{E}_{\tilde{W}_{(t,x_t)}}\Big[\max_{0\leq s\leq T} S_s \;-\; S_T\Big] \qquad (11.14)$$

Let

$$M_t \;:=\; \max_{0\leq s\leq t} S_s \qquad (11.15)$$

There is the following

Theorem 11.3: Let $c=\frac{r}{\sigma}-\frac{\sigma}{2}$ and $\kappa=\frac{2r}{\sigma^2}$. Then the price V_t, (11.14), of the lookback put (11.13) is given by

$$V_t \;=\; e^{-r\tau}M_t\times\Big[N\big(\tfrac{b_0-\tau c}{\sqrt{\tau}}\big)-\tfrac{1}{\kappa}\big(\tfrac{S_t}{M_t}\big)^{1-\kappa}N\big(-\tfrac{b_0+\tau c}{\sqrt{\tau}}\big)\Big]-S_t\times\Big[1-\tfrac{\kappa+1}{\kappa}\,N\big(-\tfrac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\big)\Big] \qquad (11.16)$$

Proof: Since $\max_{0\leq s\leq T} S_s = \max\big\{\,M_t\,,\,\max_{t\leq s\leq T} S_s\,\big\}$ and because of the above lemma we have

$$V_t \;=\; e^{-r(T-t)}\,\mathsf{E}_{\tilde{W}^{(t,T]}_{(t,x_t)}}\Big[\max\big\{M_t,\,\max_{t\leq s\leq T} S_s^{(\mu)}\big\} \;-\; S_T^{(\mu)}\Big] \qquad (11.17)$$

$$= \; e^{-r(T-t)}\,\mathsf{E}_{W^{(0,T-t]}_{(0,0)}}\Big[\max\big\{M_t,\max_{0\le s\le T-t} S_t^{(\mu)}\,e^{\sigma y_s+(r-\frac{\sigma^2}{2})s}\big\} \,-\, S_t^{(\mu)}\,e^{\sigma y_{T-t}+(r-\frac{\sigma^2}{2})(T-t)}\Big]$$
$$= \; e^{-r(T-t)}\,\mathsf{E}_{W^{(0,T-t]}_{(0,0)}}\Big[\max\big\{M_t,S_t^{(\mu)}\,e^{\sigma\max_{0\le s\le T-t}\{y_s+(\frac{r}{\sigma}-\frac{\sigma}{2})s\}}\big\} \,-\, S_t^{(\mu)}\,e^{\sigma\left(y_{T-t}+(\frac{r}{\sigma}-\frac{\sigma}{2})(T-t)\right)}\Big]$$

We abbreviate $\tau = T-t$, $S_t^{(\mu)} = S_t$ and make a Girsanov transformation $y_s + cs = v_s$, $c = \frac{r}{\sigma} - \frac{\sigma}{2}$. Then

$$\begin{aligned}
V_t \;&=\; e^{-r\tau}\int\Big[\max\big\{M_t,S_t\,e^{\sigma\max_{0\le s\le\tau}\{v_s\}}\big\} \,-\, S_t\,e^{\sigma v_\tau}\Big]\,e^{cy_\tau+\frac{c^2}{2}\tau}\,dW_{(0,0)}(\{v_s\}_{0<s\le\tau}) \\
&=\; e^{-r\tau}\int\Big[\max\big\{M_t,S_t\,e^{\sigma\max_{0\le s\le\tau}\{v_s\}}\big\} \,-\, S_t\,e^{\sigma v_\tau}\Big]\,e^{cv_\tau-\frac{c^2}{2}\tau}\,dW_{(0,0)}(\{v_s\}_{0<s\le\tau}) \\
&=\; e^{-r\tau}\int_{\mathbb{R}^2}\Big[\max\big\{M_t,S_t\,e^{\sigma b}\big\} \,-\, S_t\,e^{\sigma a}\Big]\,e^{ca-\frac{c^2}{2}\tau}\;\times \\
&\qquad\qquad \mathsf{P}\Big(v_\tau\in[a,a+da),\,\max_{0\le s\le\tau}\{v_s\}\in[b,b+db)\Big) \\
&\overset{(10.23)}{=}\; e^{-r\tau}\int_{\mathbb{R}^2}\Big[\max\big\{M_t,S_t\,e^{\sigma b}\big\} \,-\, S_t\,e^{\sigma a}\Big]\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,\chi(a<b)\,da\,db \\
&=\; e^{-r\tau}\,S_t\int_{\mathbb{R}^2}\Big[\max\big\{J_t/S_t,e^{\sigma b}\big\} \,-\, e^{\sigma a}\Big]\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,\chi(a<b)\,da\,db
\end{aligned} \tag{11.18}$$

We introduce $b_0 > 0$ according to

$$M_t/S_t = e^{\sigma b_0} \tag{11.19}$$

and write

$$\max\big\{M_t/S_t,e^{\sigma b}\big\} \;=\; M_t/S_t\,\chi(b<b_0) + e^{\sigma b}\chi(b\ge b_0) \tag{11.20}$$

This gives

$$\begin{aligned}
V_t \;&=\; e^{-r\tau}\,S_t\int_{\mathbb{R}^2}\Big[M_t/S_t\,\chi(b<b_0)+e^{\sigma b}\chi(b\ge b_0) \,-\, e^{\sigma a}\Big]\;\times \\
&\qquad\qquad e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,\chi(a<b)\,da\,db \\
&=\; e^{-r\tau}\,S_t\Big\{\int_{\mathbb{R}^2} M_t/S_t\,\chi(a<b<b_0)\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,da\,db \\
&\qquad +\int_{\mathbb{R}^2} e^{\sigma b}\chi(b\ge b_0)\,\chi(b>a)\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,da\,db \\
&\qquad -\int_{\mathbb{R}^2} e^{\sigma a}\chi(b>a)\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,da\,db\Big\} \\
&=:\; e^{-r\tau}\,S_t\Big\{I_1+I_2+I_3\Big\}
\end{aligned} \tag{11.21}$$

We first perform the b-integrals. For I_1, we obtain

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}} M_t/S_t\,\chi(a<b_0)\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{1}{\sqrt{\tau}})\,\varphi\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\Big|_a^{b_0}\,da\,db \\
&= \int_{\mathbb{R}} M_t/S_t\,\chi(a<b_0)\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{1}{\sqrt{\tau}})\Big(\varphi\left(\tfrac{2b_0-a}{\sqrt{\tau}}\right)-\varphi\left(\tfrac{a}{\sqrt{\tau}}\right)\Big)\,da\,db \\
&= M_t/S_t\int_{b_0}^{\infty} e^{c(2b_0-a')-\frac{c^2}{2}\tau}\,(-\tfrac{1}{\sqrt{\tau}})\varphi\left(\tfrac{a'}{\sqrt{\tau}}\right)\,da' \\
&\quad - M_t/S_t\int_{-\infty}^{b_0} e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{1}{\sqrt{\tau}})\varphi\left(\tfrac{a}{\sqrt{\tau}}\right)\,da \\
&= -M_t/S_t\int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} e^{c(2b_0-\sqrt{\tau}\,a)-\frac{c^2}{2}\tau}\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{a^2}{2}}\,da \\
&\quad + M_t/S_t\int_{-\infty}^{\frac{b_0}{\sqrt{\tau}}} e^{c\sqrt{\tau}\,a-\frac{c^2}{2}\tau}\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{a^2}{2}}\,da \\
&= -M_t/S_t\,e^{2cb_0}\int_{\frac{b_0}{\sqrt{\tau}}}^{\infty}\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{(a+\sqrt{\tau}\,c)^2}{2}}\,da + M_t/S_t\int_{-\infty}^{\frac{b_0}{\sqrt{\tau}}}\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{(a-\sqrt{\tau}\,c)^2}{2}}\,da \\
&= -M_t/S_t\,(M_t/S_t)^{\frac{2c}{\sigma}}\Big(1-N\big(\tfrac{b_0+\tau c}{\sqrt{\tau}}\big)\Big)+M_t/S_t\,N\big(\tfrac{b_0-\tau c}{\sqrt{\tau}}\big) \\
&= -(M_t/S_t)^{\kappa}\Big(1-N\big(\tfrac{b_0+\tau c}{\sqrt{\tau}}\big)\Big)+M_t/S_t\,N\big(\tfrac{b_0-\tau c}{\sqrt{\tau}}\big) \qquad (11.22)
\end{aligned}$$

where we introduced $\kappa=\frac{2c}{\sigma}+1=\frac{2r}{\sigma^2}$. The third term in (11.21) becomes

$$\begin{aligned}
I_3 &= -\int_{\mathbb{R}^2} e^{\sigma a}\chi(b>a)\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,da\,db \\
&= -\int_{\mathbb{R}} e^{\sigma a}\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{1}{\sqrt{\tau}})\,\varphi\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\Big|_a^{\infty}\,da \\
&= -\int_{\mathbb{R}} e^{\sigma a}\,e^{ca-\frac{c^2}{2}\tau}\,\tfrac{1}{\sqrt{\tau}}\,\varphi\left(\tfrac{a}{\sqrt{\tau}}\right)\,da \\
&= -\int_{\mathbb{R}} e^{\sigma\sqrt{\tau}\,a}\,e^{c\sqrt{\tau}\,a-\frac{c^2}{2}\tau}\,\tfrac{1}{\sqrt{2\pi}}\,e^{-\frac{a^2}{2}}da \\
&= -e^{\frac{(\sigma+c)^2}{2}\tau}e^{-\frac{c^2}{2}\tau}=-e^{r\tau} \qquad (11.23)
\end{aligned}$$

since $(\sigma+c)^2-c^2=\sigma(\sigma+2c)=\sigma(\sigma+\frac{2r}{\sigma}-\sigma)=2r$. Finally,

$$I_2 = \int_{\mathbb{R}^2} e^{\sigma b}\chi(b\geq b_0)\,\chi(b>a)\,e^{ca-\frac{c^2}{2}\tau}\,(-\tfrac{2}{\tau})\,\varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right)\,da\,db$$

$$
\begin{aligned}
&= \int_{b_0}^{\infty} db\, e^{\sigma b} \int_{-\infty}^{b} da\, e^{ca-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\tau}\right) \varphi'\left(\tfrac{2b-a}{\sqrt{\tau}}\right) \\
&= \int_{b_0}^{\infty} db\, e^{\sigma b} \int_{b}^{\infty} da\, e^{c(2b-a)-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\tau}\right) \varphi'\left(\tfrac{a}{\sqrt{\tau}}\right) \\
&= \int_{b_0}^{\infty} db\, \underbrace{e^{(\sigma+2c)b}}_{=u'} \underbrace{\int_{b}^{\infty} da\, e^{-ca-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\tau}\right) \varphi'\left(\tfrac{a}{\sqrt{\tau}}\right)}_{=v} \\
&= \tfrac{1}{\sigma+2c}\, e^{(\sigma+2c)b} \int_{b}^{\infty} da\, e^{-ca-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\tau}\right) \varphi'\left(\tfrac{a}{\sqrt{\tau}}\right) \Big|_{b_0}^{\infty} \\
&\quad + \int_{b_0}^{\infty} db\, \tfrac{1}{\sigma+2c}\, e^{(\sigma+2c)b} e^{-cb-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\tau}\right) \varphi'\left(\tfrac{b}{\sqrt{\tau}}\right) \\
&= -\tfrac{1}{\sigma+2c}\, e^{(\sigma+2c)b_0} \int_{b_0}^{\infty} da\, e^{-ca-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\tau}\right) \varphi'\left(\tfrac{a}{\sqrt{\tau}}\right) \\
&\quad + \int_{b_0}^{\infty} db\, \tfrac{1}{\sigma+2c}\, e^{(\sigma+c)b} e^{-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\tau}\right) \varphi'\left(\tfrac{b}{\sqrt{\tau}}\right) \\
&= -\tfrac{1}{\sigma+2c}\, e^{(\sigma+2c)b_0} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da\, e^{-\sqrt{\tau}\,ca-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\sqrt{\tau}}\right) \varphi'(a) \\
&\quad + \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} db\, \tfrac{1}{\sigma+2c}\, e^{(\sigma+c)\sqrt{\tau}\,b} e^{-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\sqrt{\tau}}\right) \varphi'(b) \qquad (11.24)
\end{aligned}
$$

Because of

$$
\begin{aligned}
&\int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da\, e^{-\sqrt{\tau}\,ca-\frac{c^2}{2}\tau} \left(-\tfrac{2}{\sqrt{\tau}}\right) \varphi'(a) \\
&= \tfrac{2}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da\, a\, e^{-\sqrt{\tau}\,ca-\frac{c^2}{2}\tau} e^{-\frac{a^2}{2}} \\
&= \tfrac{2}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da\, a\, e^{-\frac{(a+\sqrt{\tau}\,c)^2}{2}} \\
&= \tfrac{2}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da\, (a+\sqrt{\tau}\,c) e^{-\frac{(a+\sqrt{\tau}\,c)^2}{2}} - \tfrac{2\sqrt{\tau}\,c}{\sqrt{2\pi\tau}} \int_{\frac{b_0}{\sqrt{\tau}}}^{\infty} da\, e^{-\frac{(a+\sqrt{\tau}\,c)^2}{2}} \\
&= \tfrac{2}{\sqrt{2\pi\tau}}\, e^{-\frac{(b_0+\tau c)^2}{2\tau}} - 2c\Big(1 - N\big(\tfrac{b_0+\tau c}{\sqrt{\tau}}\big)\Big) \qquad (11.25)
\end{aligned}
$$

(11.24) becomes

$$
\begin{aligned}
I_2 &= -\tfrac{1}{\sigma+2c}\, e^{(\sigma+2c)b_0} \Big(\tfrac{2}{\sqrt{2\pi\tau}}\, e^{-\frac{(b_0+\tau c)^2}{2\tau}} - 2c + 2cN\big(\tfrac{b_0+\tau c}{\sqrt{\tau}}\big)\Big) \\
&\quad + \tfrac{1}{\sigma+2c}\, e^{\frac{(\sigma+c)^2-c^2}{2}\tau} \Big(\tfrac{2}{\sqrt{2\pi\tau}}\, e^{-\frac{(b_0-\tau(c+\sigma))^2}{2\tau}} + 2(c+\sigma) - 2(c+\sigma)N\big(\tfrac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\big)\Big) \\
&= \tfrac{2c}{\sigma+2c}\, e^{(\sigma+2c)b_0}\Big(1 - N\big(\tfrac{b_0+\tau c}{\sqrt{\tau}}\big)\Big) + \tfrac{2c+2\sigma}{\sigma+2c}\, e^{r\tau}\Big(1 - N\big(\tfrac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\big)\Big) \qquad (11.26)
\end{aligned}
$$

Substituting this in (11.21), we arrive at

$$\begin{aligned} V_t &= e^{-r\tau} S_t \Big\{ -(M_t/S_t)^\kappa N\big(-\tfrac{b_0+\tau c}{\sqrt{\tau}}\big) + M_t/S_t\, N\big(\tfrac{b_0-\tau c}{\sqrt{\tau}}\big) + \tfrac{2c}{\sigma+2c}\, e^{(\sigma+2c)b_0} N\big(-\tfrac{b_0+\tau c}{\sqrt{\tau}}\big) \\ &\qquad + \tfrac{2c+2\sigma}{\sigma+2c}\, e^{r\tau} N\big(-\tfrac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\big) - e^{r\tau} \Big\} \\ &= e^{-r\tau} M_t \Big[N\big(\tfrac{b_0-\tau c}{\sqrt{\tau}}\big) - \tfrac{1}{\kappa}(M_t/S_t)^{\kappa-1} N\big(-\tfrac{b_0+\tau c}{\sqrt{\tau}}\big) \Big] - S_t \Big[1 - \big(1+\tfrac{1}{\kappa}\big) N\big(-\tfrac{b_0-\tau(c+\sigma)}{\sqrt{\tau}}\big) \Big] \end{aligned}$$

and the theorem is proven. ■