Appendix 1

A1: American Options in the Binomial Model

So far we were dealing with options which can be excercised only at a fixed time, at their maturity date T. These are european options. In a complete model, like the Binomial or the Black-Scholes model, these options can be replicated exactly. Now we are considering options which can be excercised at an arbitrary time $t \in [0, T]$. These are called american options. Thus, if a bank is selling an american option, it has to be prepared to pay the payoff to the customer not only at $t = T$, but at an arbitrary $t \in [0, T]$. Therefore it would be desirable to replicate the option for all $t \in [0, T]$, not only at $t = T$. However, this is not possible. But it is possible to set up a selffinancing strategy whose portfolio value is always greater or equal to the option's payoff and this leads to an arbitrage free fair price for the american option contract. We first consider the situation in discrete time, in the Binomial model.

Let

$$S_k = S_0 \prod_{j=1}^{k} X_j \tag{1.1}$$

be the price process for the Binomial model. That is, the X_j are identically independently distributed with

$$X_j = \begin{cases} U = 1 + u & \text{with probability } p \\ D = 1 + d & \text{with probability } q = 1 - p \end{cases} \tag{1.2}$$

The discounted price process $s_k = R^{-k} S_k$, $R = 1 + r$, is given by

$$s_k = s_0 \prod_{j=1}^{k} x_j \tag{1.3}$$

where the $x_j = X_j / R$ have the distribution

$$\begin{aligned} dB(\{x_j\}) &= \prod_{j=1}^{N} db(x_j) && (1.4) \\ db(x) &= \big\{ p\, \delta(x - U/R) + (1-p)\, \delta(x - D/R) \big\} dx && (1.5) \end{aligned}$$

Let C be the payoff of some (exotic or not) american option. That is, excercising the option at time $k \in \{1, 2, ..., N\}$ gives a payment of

$$C = C\big(S_0, ..., S_k\big) \tag{1.6}$$

For a simple call or put (1.6) reduces to $C = C(S_k)$. Let

$$c_k = R^{-k} C(S_0, ..., S_k) \tag{1.7}$$

be the discounted payoff at time k. We are looking for some selffinancing strategy (δ_k) whose discounted portfolio value $v_k = R^{-k} V_k$ at time k is always bigger than c_k,

$$v_k \;=\; v_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1}) \;\stackrel{!}{\geq}\; c_k \tag{1.8}$$

At time $k = N$, we should have $v_N = c_N$. At time $k = N-1$, v_{N-1} should be such that we can guarantie $v_N = c_N$ at $k = N$ and it must be at least c_{N-1}. In the first lecture on the Binomial model and in lecture 5 we saw that we can guarantie $v_N = c_N$ at $k = N$ if we define

$$\begin{aligned} v_{N-1} &= \tilde{p}\, v_N(s_0, ..., s_{N-1}, s_{N-1}U/R) + (1-\tilde{p})\, v_N(s_0, ..., s_{N-1}, s_{N-1}D/R) \\ &= \mathsf{E}_{\tilde{B}}[v_N \,|\, s_{N-1}] \end{aligned} \tag{1.9}$$

where $\tilde{p} = \frac{R-D}{U-D}$ is the probability of the equivalent martingale measure $d\tilde{B}$. That is, if we define a sequence of discounted portfolio values $(u_k)_{0\leq k\leq N}$ inductively by

$$u_N \;=\; c_N \tag{1.10}$$

$$u_{k-1} \;=\; \max\big\{c_{k-1}, \mathsf{E}_{\tilde{B}}[u_k \,|\, s_{k-1}]\big\}, \quad k = N, N-1, ..., 1 \tag{1.11}$$

then we must have

$$v_k \geq u_k \quad \forall k = 0, ..., N \tag{1.12}$$

Observe that we cannot write equal in (1.12) since it is not clear that the portfolio values u_k defined in (1.10,1.11) can be generated by some selffinancing strategy. In fact, in general they cannot. Thus we have to find 'the smallest' selffinancing strategy, which has the lowest price v_0 and satisfies (1.12). It is obtained as follows.

Theorem 8.1: Let $(u_k)_{0\leq k\leq N}$ be the Snell envelope of the discounted american claim $(c_k)_{0\leq k\leq N}$ defined by (1.10,1.11). For $k = 1, ..., N$, let

$$\delta_{k-1}(s_0, ..., s_{k-1}) \;=\; \frac{u_k(s_0, ..., s_{k-1}, s_{k-1}U/R) - u_k(s_0, ..., s_{k-1}, s_{k-1}D/R)}{s_{k-1}U/R - s_{k-1}D/R} \tag{1.13}$$

and let

$$v_k = u_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1}) \tag{1.14}$$

be the discounted portfolio value of the selffinancing strategy defined by (1.13) with initial price u_0. Then any selffinancing strategy $\tilde{v}_k = \tilde{v}_0 + \sum_{j=1}^{k} \tilde{\delta}_{j-1}(s_j - s_{j-1})$ with $\tilde{v}_k \geq c_k$ for all k satisfies $\tilde{v}_k \geq u_k$ for all k. In particular, each such strategy has a larger price than (1.14), $\tilde{v}_0 \geq u_0$. The relation between (u_k) and (v_k) is given by

$$v_k \; = \; u_k + \sum_{j=1}^{k} \Big(\max\big\{ c_{j-1}, \tilde{u}_{j-1} \big\} \; - \; \tilde{u}_{j-1} \Big) \; \geq \; u_k \tag{1.15}$$

where $\tilde{u}_{k-1} := \mathsf{E}_{\tilde{B}}[u_k \,|\, s_{k-1}]$.

Proof: We first show (1.15). Recall from Theorem 1.1 that if δ_{k-1} is given by (1.13) and if $\tilde{u}_{k-1}$ is given by

$$\begin{aligned} \tilde{u}_{k-1} \;&=\; \tilde{p}\, u_k(s_0, ..., s_{k-1}, s_{k-1}U/R) + (1-\tilde{p})\, u_k(s_0, ..., s_{k-1}, s_{k-1}D/R) \\ &=\; \mathsf{E}_{\tilde{B}}[u_k \,|\, s_{k-1}] \end{aligned} \tag{1.16}$$

then there is the relation

$$\tilde{u}_{k-1} + \delta_{k-1}(s_k - s_{k-1}) = u_k \tag{1.17}$$

Hence,

$$\begin{aligned} v_k \quad &= \quad u_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1}) \\ &\overset{(1.17)}{=} \quad u_0 + \sum_{j=1}^{k} \big(u_j - \tilde{u}_{j-1} \big) \\ &= \quad u_k + \sum_{j=1}^{k} \big(u_{j-1} - \tilde{u}_{j-1} \big) \\ &\overset{(1.11)}{=} \quad u_k + \sum_{j=1}^{k} \Big(\max\big\{ c_{j-1}, \tilde{u}_{k-1} \big\} \; - \; \tilde{u}_{k-1} \Big) \end{aligned} \tag{1.18}$$

which proves (1.15). The inequalities $\tilde{v}_k \geq u_k$ for all k can be obtained by induction. For $k = N$, $\tilde{v}_N \geq c_N = u_N$. Suppose $\tilde{v}_k \geq u_k$ holds for k. Then

$$\tilde{v}_{k-1} \;\; = \;\; \mathsf{E}_{\tilde{B}}[\tilde{v}_k | s_{k-1}] \;\; \geq \;\; \mathsf{E}_{\tilde{B}}[u_k | s_{k-1}] \;\; = \;\; \tilde{u}_{k-1} \tag{1.19}$$

Since also by assumption $\tilde{v}_{k-1} \geq c_{k-1}$ we have $\tilde{v}_{k-1} \geq \max\{c_{k-1}, \tilde{u}_{k-1}\} = u_{k-1}$ which completes the induction. ■

Theorem 8.1 solves the hedging problem for american options in the Binomial model and, by approximation with small Δt, also for the Black-Scholes model. One has to compute the Snell envelope (u_k) which can be easily done on an excel-sheet using the definition (1.10,1.11) and then the delta's for a selffinancing strategy are given by (1.13). Another characterization of the sequence (u_k), which is probably of more theoretical interest, can be given in terms of stopping times. It reads as follows.

Theorem 8.2: Let $\mathcal{T}_k = \{\tau : \{s_j\}_{0\leq j\leq N} \to \tau(\{s_j\}) \in \{k, k+1, ..., N\}, \tau \text{ stopping time}\}$ be the set of all stopping times bigger or equal than k. Let $c_k = R^{-k}C(S_0, ..., S_k)$ be the discounted payoff of some american option and let (u_k) be the Snell envelope defined by (1.10,1.11). Then

$$\begin{aligned} u_k &= \sup_{\tau\in\mathcal{T}_k} \mathsf{E}_{\tilde{B}}[\, c_\tau \,|\, s_k \,] && (1.20)\\ &= \mathsf{E}_{\tilde{B}}\big[\, c_{\tau^k_{u=c}} \,|\, s_k \,\big] && (1.21)\end{aligned}$$

where $\tau^k_{u=c}$ is the stopping time defined by

$$\tau^k_{u=c} = \min\{j \geq k \,|\, u_j = c_j\} \tag{1.22}$$

Proof: From (1.15) we have

$$u_k = v_k - \sum_{j=1}^{k}\Big(\max\{c_{j-1}, \tilde{u}_{j-1}\} - \tilde{u}_{j-1}\Big) \tag{1.23}$$

and for $m \geq k$

$$\begin{aligned} u_m &= u_k + (u_m - u_k)\\ &= u_k + v_m - v_k - \sum_{j=k+1}^{m}\Big(\max\{c_{j-1}, \tilde{u}_{j-1}\} - \tilde{u}_{j-1}\Big) && (1.24)\end{aligned}$$

In particular,

$$\begin{aligned} u_{\tau^k_{u=c}} &= u_k + v_{\tau^k_{u=c}} - v_k - \sum_{j=k+1}^{\tau^k_{u=c}}\Big(\max\{c_{j-1}, \tilde{u}_{j-1}\} - \tilde{u}_{j-1}\Big)\\ &= u_k + v_{\tau^k_{u=c}} - v_k - \sum_{j=k+1}^{\tau^k_{u=c}}\Big(\tilde{u}_{j-1} - \tilde{u}_{j-1}\Big)\\ &= u_k + v_{\tau^k_{u=c}} - v_k && (1.25)\end{aligned}$$

Thus,

$$\begin{aligned}
\mathsf{E}_{\tilde{B}}\big[\, c_{\tau^k_{u=c}} \,|\, s_k \,\big] &= \mathsf{E}_{\tilde{B}}\big[\, u_{\tau^k_{u=c}} \,|\, s_k \,\big] \\
&= u_k + \mathsf{E}_{\tilde{B}}\big[\, v_{\tau^k_{u=c}} \,|\, s_k \,\big] - v_k \\
&\overset{\text{Lemma 8.3}}{=} u_k
\end{aligned} \tag{1.26}$$

which proves (1.21). Furthermore, for every stopping time $\tau \in \mathcal{T}_k$,

$$\begin{aligned}
c_\tau &\leq u_\tau \\
&= u_k + v_\tau - v_k - \sum_{j=k+1}^{\tau} \Big(\max\big\{c_{j-1}, \tilde{u}_{j-1}\big\} \;-\; \tilde{u}_{j-1}\Big)
\end{aligned} \tag{1.27}$$

and therefore, using Lemma 8.3 below again,

$$\begin{aligned}
\mathsf{E}_{\tilde{B}}[\, c_\tau \,|\, s_k \,] &\leq u_k + \underbrace{\mathsf{E}_{\tilde{B}}[\, v_\tau - v_k \,|\, s_k \,]}_{=0} - \sum_{j=k+1}^{N} \underbrace{\mathsf{E}_{\tilde{B}}\Big[\, \big(\max\big\{c_{j-1}, \tilde{u}_{j-1}\big\} \;-\; \tilde{u}_{j-1}\big)\chi(j \leq \tau) \,|\, s_k \,\Big]}_{\geq 0} \\
&\leq u_k
\end{aligned} \tag{1.28}$$

This implies also $\sup_{\tau \in \mathcal{T}_k} \mathsf{E}_{\tilde{B}}[\, c_\tau \,|\, s_k \,] \leq u_k$ and, because of (1.26), the equal sign follows. ■

Lemma 8.3 (Stopping Theorem): Let $v_k = v_0 + \sum_{j=1}^{k} \delta_{j-1}(s_j - s_{j-1})$ be a selffinancing strategy and let τ be a stopping time. Let $\tilde{\mathsf{E}}$ denote the expectation with respect to the martingale measure. Then $\tilde{\mathsf{E}}[v_\tau] = v_0$, and, for $\tau \geq k$,

$$\tilde{\mathsf{E}}[v_\tau | s_k] = v_k \tag{1.29}$$

Proof: For $\tau \geq k$ we have

$$\begin{aligned}
\tilde{\mathsf{E}}[v_\tau | s_k] &= v_k + \sum_{j=k+1}^{N} \tilde{\mathsf{E}}\big[\delta_{j-1}(s_j - s_{j-1})\chi(j \leq \tau) \,|\, s_k\big] \\
&= v_k + \sum_{j=k+1}^{N} \tilde{\mathsf{E}}\big[\delta_{j-1}(s_j - s_{j-1}) \,|\, s_k\big] - \sum_{j=k+1}^{N} \tilde{\mathsf{E}}\big[\delta_{j-1}(s_j - s_{j-1})\chi(j > \tau) \,|\, s_k\big] \\
&= v_k - \sum_{j=k+1}^{N} \tilde{\mathsf{E}}\big[\delta_{j-1}(s_j - s_{j-1})\chi(j > \tau) \,|\, s_k\big]
\end{aligned} \tag{1.30}$$

Since τ is a stopping time, we have $\tau = \tau(\{s_m\}_{0 \leq m \leq \tau})$. Thus, for $\tau < j$, τ is a function of at most $s_0, s_1, ..., s_{j-1}$. But then we can move the integrals over $s_n, ..., s_j$ directly to the

s_j in the last line of (1.30) to obtain

$$\begin{aligned}
\tilde{\mathsf{E}}[v_\tau|s_k] &= v_k - \sum_{j=k+1}^{N} \tilde{\mathsf{E}}\big[\delta_{j-1}\underbrace{\big(\tilde{\mathsf{E}}[s_j|s_{j-1}] - s_{j-1}\big)}_{=0}\chi(j>\tau)\,|\,s_k\big] \\
&= v_k \qquad (1.31)
\end{aligned}$$

since (s_j) is a martingale with respect to $\tilde{\mathsf{E}}$. ■

The next lemma states that if the option value of some european option is always bigger than the option's payoff, $V_k(S_k) \geq C(S_k)$, then the Snell envelope coincides with the discounted V_k.

Lemma 8.4: Let V_k be the portfolio value of a replicating strategy for some european option with payoff $C(S_0, ..., S_N)$, that is, V_k is the option value at time k. Then, if

$$V_k(S_0, ..., S_k) \;\geq\; C(S_0, ..., S_k) \quad \forall k = N, N-1, ..., m \qquad (1.32)$$

the undiscounted Snell envelope $U_k = R^k u_k$ satisfies

$$U_k \;=\; V_k \quad \forall k = N, N-1, ..., m. \qquad (1.33)$$

In particular, if for $c_k = R^{-k}C(S_k)$

$$\mathsf{E}_{\tilde{B}}[c_k|s_{k-1}] \;\geq\; c_{k-1} \quad \forall k \qquad (1.34)$$

then $v_k \geq c_k$ and $u_k = v_k$ for all k.

Proof: We prove (1.33) by induction on k. For $k = N$, $u_N = c_N = v_N$. Suppose (1.33) holds for k. Observe that since $v_k = v_{k-1} + \delta_{k-1}(s_k - s_{k-1})$ we have $v_{k-1} = \mathsf{E}_{\tilde{B}}[v_k|s_{k-1}]$. Then

$$\begin{aligned}
u_{k-1} &= \max\big\{c_{k-1}, \mathsf{E}_{\tilde{B}}[u_k|s_{k-1}]\big\} \\
&= \max\big\{c_{k-1}, \mathsf{E}_{\tilde{B}}[v_k|s_{k-1}]\big\} \\
&= \max\big\{c_{k-1}, v_{k-1}]\big\} \\
&= v_{k-1} \qquad (1.35)
\end{aligned}$$

if $v_{k-1} \geq c_{k-1}$. This proves (1.33). (1.34) follows similar. By induction, starting at $k = N$ with $v_N = c_N$,

$$\begin{aligned}
v_{k-1} &= \mathsf{E}_{\tilde{B}}[v_k|s_{k-1}] \\
&\geq \mathsf{E}_{\tilde{B}}[c_k|s_{k-1}] \\
&\geq c_{k-1} \qquad (1.36)
\end{aligned}$$

and the lemma is proven ■

Condition (1.34) holds for a call option, but not for a put. Namely, for a call

$$c_k = R^{-k}(S_k - K)_+ = (s_k - K/R^k)_+ = (s_{k-1}x_k - K/R^k)_+ \tag{1.37}$$

Since $f(x) = (s_{k-1}x - K/R^k)_+$ is a convex function, we can apply Jensen's inequality, $\int f(x)\,dP(x) \geq f\big(\int x\,dP(x)\big)$ for any probability measure dP and convex function f, to obtain

$$\begin{aligned} \mathsf{E}_{\tilde{B}}[c_k|s_{k-1}] &= \int \big(s_{k-1}x_k - K/R^k\big)_+\, d\tilde{b}(x_k) \\ &\geq \Big(\textstyle\int s_{k-1}x_k\, d\tilde{b}(x_k) - K/R^k\Big)_+ \\ &= (s_{k-1} - K/R^k)_+ \\ &\stackrel{R\geq 1}{\geq} (s_{k-1} - K/R^{k-1})_+ = c_{k-1} \end{aligned} \tag{1.38}$$

If one tries the same computation for a put, one ends up with

$$\begin{aligned} \mathsf{E}_{\tilde{B}}[c_k|s_{k-1}] &= \int \big(K/R^k - s_{k-1}x_k\big)_+\, d\tilde{b}(x_k) \\ &\geq \Big(K/R^k - \textstyle\int s_{k-1}x_k\, d\tilde{b}(x_k)\Big)_+ \\ &= (K/R^k - s_{k-1})_+ \end{aligned} \tag{1.39}$$

but this cannot be estimated against $(K/R^{k-1} - s_{k-1})_+$ for $R \geq 1$. Thus we can summarize

Corollary 8.5: Let the discounting factor $R \geq 1$. Then the values of american and european calls coincide but the values of american and european puts differ. The value of an american put at time k is given by the undiscounted Snell envelope $U_k = R^k u_k$ defined by (1.10,1.11).